A Minimally Algebraic Solution to a Famous *Sangaku* Problem

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PROBLEM:

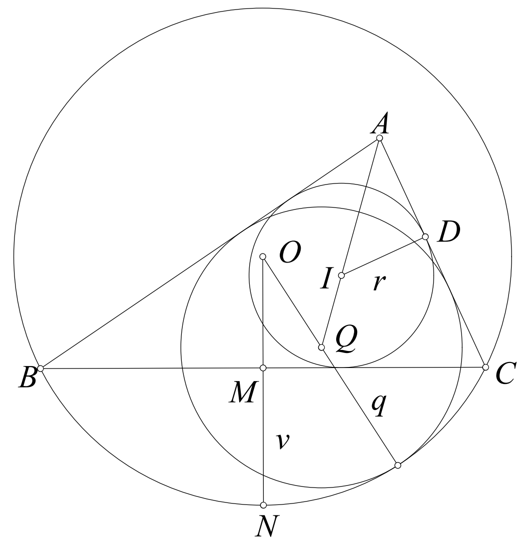

Figure 1

Let the semiperimeter of triangle *ABC* with inradius *r* (below left) be *s*, and the sagitta to side *BC* be *v*. Circle *(O)* passes through *B* and *C*. Let circle *(Q)q* be tangent to *AB*, *AC*, and *(O)* internally (Figure 1). Prove that $q = r + \dfrac{2v(s-b)(s-c)}{as}$.[1]

SOLUTION:

If δ is the angle $CBN = BCN$, then $\dfrac{2rv(s-b)(s-c)}{as} = \tan\dfrac{\alpha}{2}\tan\delta$ because

(1) $\tan\dfrac{\alpha}{2} = \dfrac{r}{s-a}$ (in right triangle *AID*),

(2) $\tan\delta = \dfrac{v}{a/2}$ (in right triangles *BMN* and *CMN*),[2] and (3) by Heron's formula in the form $r^2 s = (s-a)(s-b)(s-c)$, $\dfrac{2rv(s-b)(s-c)}{as} = \dfrac{2v}{a}\cdot\dfrac{r^2}{s-a}$. Thus the problem is equivalent to proving that $q - r = r\tan\dfrac{\alpha}{2}\tan\delta$.

Let *E* be the point where *(Q)* touches *AC*, and *F* be the foot of the perpendicular from *I* to *EQ* (Figure 2). Then *DI* || *EQ* and *IEQ* = *DIE*. We immediately have *EQ* – *EF* = *q* – *r* = *FQ* = *IF* tan (α/2) = (*EF* tan *IEF*) tan (α/2) = *r* tan δ tan (α/2), but only if *IEQ* = δ.

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[1] Fukagawa & Pedhoe 1989, 2.2.8 (1781, n.pl.), "a hard but important problem."

[2] Fukagawa & Rigby 2002 (p. 97) state this much.

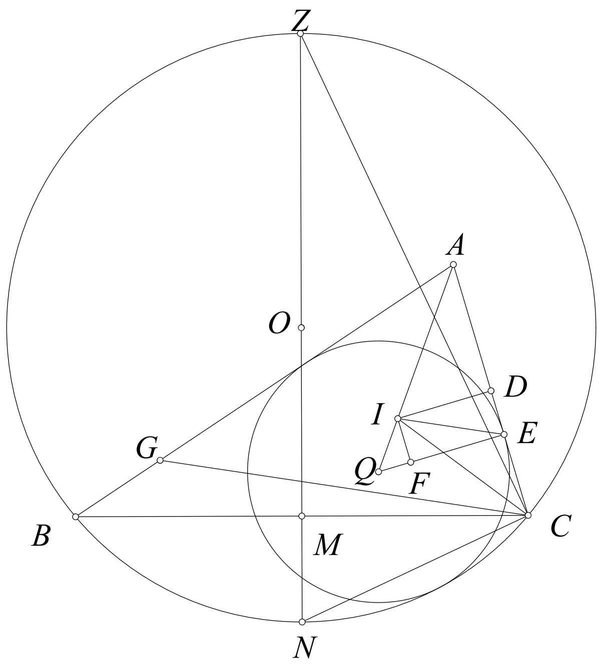

Figure 2

On the other hand, say there is an *E′* on *AC*, *E′Q′* ⊥ *AC*, such that *IE′Q′* = *δ*. Then *AE′I* is complementary to *δ*. If we construct *CG* || *E′I*, *AE′I* = *ACG*. Moreover, if *OM* cuts *(O)* at *N* and *Z*, then *δ* = *CZN*, so *BCZ* is complementary to *δ* and *ACG* = *BCZ*. Therefore *ACZ* = *BCG*. That is, if *IE′Q′* = *δ*, then *CZ* and *CG* are isogonal with respect to angle *ACB*.

This gives us a way to construct the points *E* and *Q* given just *(O)* and *ABC:* reflect *CZ* in *CI* and draw the line parallel to the reflection through *I*, marking its intersection with *AC* as *E*. The perpendicular to *AC* from *E* cuts *AI* in *Q*. Now *IEQ* = *δ* by construction. Since *I* is determined by *ABC*, these are the same *E* and *Q* defined in the "only if" case. □

COMPANION PROBLEM:

If *A* is outside *(O)* (Figure 3), then, in addition, $x = r - \dfrac{a(s-b)(s-c)}{2vs}$.[3]

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[3] Fukagawa & Rigby 2002 (p. 32) incorrectly write $x = r - \dfrac{2v(s-b)(s-c)}{as}$.

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SOLUTION:

Note that $\frac{ra(s-b)(s-c)}{2vs} = \tan\frac{\alpha}{2}\cot\delta$ for the reasons previously stated. Thus the companion problem is equivalent to proving that $r - x = r\tan\frac{\alpha}{2}\cot\delta$.

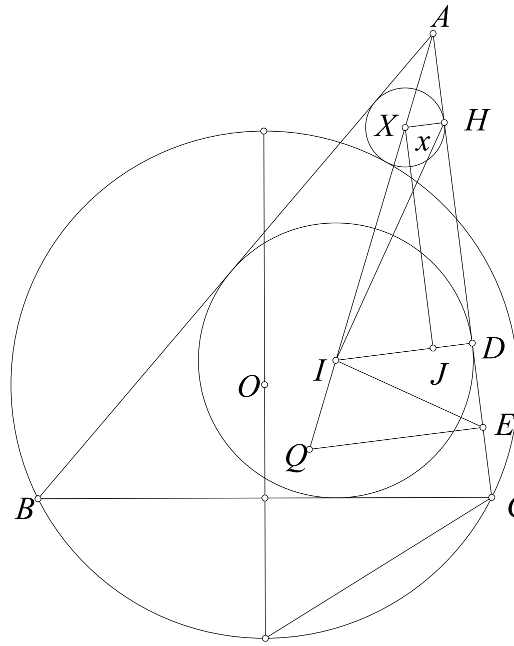

Figure 3

Given *(X)* tangent to *(O)* externally, to *AB*, and to *AC* at *H*, we have *DI* – *HQ* = *r* – *x* = *JX* tan (*α*/2) = *HD* tan (*α*/2) = *r* cot *δ* tan (*α*/2), but only if *DHI* = *δ*. On the other hand, if *DH'I* = *δ* for some arbitrary *H'* on *AC*, then *H'I* must be perpencidular to the *EI* previously constructed because *DI* ⊥ *AC* and *DIE* = *δ*. So we construct *E* as before, draw the perpendicular to *EI* through *I*, and mark its intersection with *AC* as *H*. The perpendicular to *AC* through *H* cuts *AI* in *X*. Since *DHI* = *δ* by construction and *I* is fixed, *H* and *X* are the same points defined for the "only if" case. □

Remarks:

1. Even when *A* is outside *(O)*, we still have *q* – *r* = *r* tan *δ* tan (*α*/2) for *(Q)* internally tangent to *(O)*. It is not generally true that *r* – *x* = *q* – *r*. Apparently, when the "tan" was corrected to "cot" on p. 32 of Fukagawa & Rigby 2002 (the "cot" is set irregularly in italics), the corresponding correction in the expression to its left was missed.

2. Notice that the foregoing proofs do not require the Sawayama Lemma.[4]

3. Fukagawa and Rigby attribute both problems to Ajima. They sketch what they say is the traditional solution of the first problem, developing through several steps the quadratic equation

$$2avs(s-a)q^2 + \left[\frac{a^2\Delta}{s-a} - \frac{4v^2\Delta}{s} - 2av(b+c)\right]\Delta q + \left[2av - \frac{2\Delta^2 av}{s^2(s-a)^2} - \frac{a^2\Delta}{s(s-a)} + \frac{4v^2\Delta}{s(s-a)}\right]\Delta^2 = 0$$

(where Δ is the area of *ABC*). They assert that this reduces to $q = r + \frac{2v(s-b)(s-c)}{as}$ and

[4] Ayme (2003). Y. Sawayama, an instructor at the Central Military School in Tōkyō published the lemma in 1905 coincidental to solving another problem.

that the solution of the companion “is similar.” Without doubting these claims, I find it hard to believe that Ajima used such brute-force algebraic methods as a discovery procedure. Although he probably would have thought in terms of ratios of sides of similar right triangles instead of “tan $\alpha/2$” and “tan δ” or “cot δ,” and may have felt that an algebraic “formalization” was necessary, he surely must have been guided by the realization that, in both cases, δ shows up in unexpected places.

References

Ayme, Jean-Louis. 2003. Sawayama and Thébault’s Theorem, *Forum Geometricorum* 3.225–29.

Fukagawa, Hidetoshi, and Dan Pedhoe. 1989. *Japanese temple geometry problems*. Winnipeg: Charles Babbage Research Centre.

———, and J. F. Rigby. 2002. Traditional Japanese mathematics problems of the 18th and 19th Centuries. Singapore: SCT Press.