Romania TST 2015

Lemma : Let h_A, h_B, h_C denote the lenghts of altitudes from A, B and C respectivelly. Then

$$\frac{1}{h_A}+\frac{1}{h_B}+\frac{1}{h_C}=\frac{1}{r}.$$

It is a well known result, hence it will not be proved.

Proof 1 (Leo Giugiuc) : It is suffice to show that $\frac{1}{R_A}\le\frac{2}{h_A}$. Let O denote the circumcenter of ABC and R the circumradius of ABC. We choose $A(0,2)$, $B(-2b,0)$, $C(2c,0)$ $(b=\cot B,\ c=\cot C)$;

I've been shown before that $O(c-b,1-bc)$ and $R=\sqrt{(1+b^2)(1+c^2)}$. Since (O_A) and (O) are internally tangent in A, then A, O_A, O are collinear and the vectors $\overrightarrow{AO_A}, \overrightarrow{AO}$ have the same orientation hence $\exists\, k>0$ such that $O_A\big(k(c-b),2-k(1+bc)\big)$. Let A' be the tangency point btw (O_A) and BC; then $AO_A{}^2=A'O_A{}^2=R_A^2 \Longrightarrow k^2(1+b^2)(1+c^2)=[2-k(1+bc)]^2 \Longrightarrow (c-b)^2k^2+$

$$+4(1+bc)k-4=0 \Longrightarrow k=\begin{cases} 2\cdot\dfrac{-(1+bc)+\sqrt{(1+b^2)(1+c^2)}}{(c-b)^2}, & if\ c\neq b \\ \dfrac{1}{1+b^2}, & if\ c=b \end{cases}.\ In\ conclusion$$

$$\frac{1}{R_A}=\frac{1}{2}\cdot\frac{(c-b)^2}{(1+b^2)(1+c^2)-(1+bc)\sqrt{(1+b^2)(1+c^2)}}\ \ if\ c\neq b\ and\ \frac{2}{h_A}=1.$$ We'll prove that

$$\frac{(c-b)^2}{2}\le(1+b^2)(1+c^2)-(1+bc)\sqrt{(1+b^2)(1+c^2)}\leftrightarrow\left[\sqrt{(1+b^2)(1+c^2)}-(1+bc)\right]^2\ge 0,$$ whic is true. Since $c\neq b$ the inequaliy is strictly. If $c=b$ then $O_A(0,1)$ and $R_A=1\Longrightarrow\frac{1}{R_A}=\frac{2}{h_A}$.

Due to the simetry the other two inequalities will be true too. By adding up we get the desired relation. Equality does hold iff ΔABC is equilateral.

Second proof is due to a Romanian student during the contest (I don't know his name).

By triangle's inequality, $AO_A+A'O_A\ge AA'\Longrightarrow 2R_A\ge AA'$; but $AA'\ge h_A\Longrightarrow\frac{1}{R_A}\le\frac{2}{h_A}$. Done!

Leo