Construct a triangle, given c_A, c_B, d_C

The identity

$$ab \;=\; d_C^2 + c_A c_B \tag{1}$$

leads to a simple solution of the folowing construction problem: construct a triangle ABC given c_A, c_B, and d_C (Figure 1). Since we know that $b/a = c_A/c_B$, using (1) we

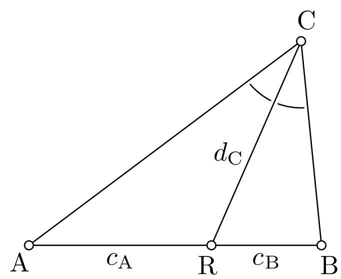


Figure 1

immediately get

$$a^2 \;=\; (a/b)ab \;=\; (c_B/c_A)\big(c_A c_B + d_C^2\big) \;=\; c_B\big(c_B + d_C^2/c_A\big)\ , \tag{2}$$

and similarly

$$b^2 \;=\; c_A\big(c_A + d_C^2/c_B\big)\ . \tag{3}$$

And that's it. We could easily read a triangle's construction off the formulas for a and b. But wait: shouldn't we ask ourselves if we have really obtained a triangle? We should, and we shall. The tentative solution a, b, $c = c_A + c_B$ represents the sides of a triangle if and only if each of a, b, c is less than the sum of the other two. Since clearly $a > c_A$ and $b > c_B$, the inequality $c < a + b$ is satisfied. The other two inequalities $a < b + c$ and $b < a + c$ can be wrapped together into the single inequality $|a - b| < c$, which is equivalent to the inequality $a^2 - 2ab + b^2 = (a-b)^2 < c^2$. We substitute the expressions for a^2 from (2), for ab from (1), for b^2 from (3), and replace c with $c_A + c_B$, to obtain, after some rearrangements, the equivalent inequality $\big((c_A - c_B)d_C\big)^2 < (2c_A c_B)^2$, that is, $|c_A - c_B| \cdot d_C < 2c_A c_B$. This condition is certainly satisfied if $c_A = c_B$: in this special case the triangle ABC is isosceles and can have the bisector = altitude CR of any length we desire. Otherwise, when $c_A \neq c_B$, there is an upper bound for d_C,

$$d_C \;<\; \frac{2\,c_A c_B}{|c_A - c_B|}\ ; \tag{4}$$

if the given c_A, c_B, d_C fail to satisfy this inequality, then there is no (non-degenerate) triangle having the prescribed c_A, c_B, d_C, and the construction problem has no solution.

We have obtained the condition (4) through computations. Now we are going to solve our construction problem as genuine geometers do it, aproaching it synthetically, as they would say. This way we will get to understand the geometric meaning of the condition (4), and as a bonus obtain a simple construction of the required triangle.

We assume, for the sake of definiteness, that $c_{\mathrm{A}} > c_{\mathrm{B}}$, and we let A, B, R be points on a line, with the point R between the points A and B, so that $\mathrm{AR} = c_{\mathrm{A}}$ and $\mathrm{BR} = c_{\mathrm{B}}$, as in Figure 2. For a while we shove aside the required length d_{C} of the bisector; we will

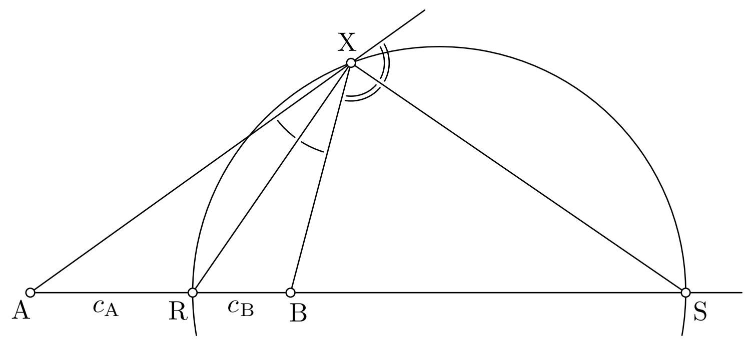


Figure 2

drag it back into the fray when the time comes.

Let X be any point in the plane, not on the line AB, such that in the triangle ABX the bisector of the internal angle at X is XR. (In Figure 2 the point X is above the line AB; this will not affect the generality of our discussion since a situation with a point X below the line is the reflection in the line AB af a situation with a point X above the line.) Let XS be the bisector of the external angle at X, with the end-point S on the line AB. Since $\mathrm{AS} : \mathrm{BS} = \mathrm{AX} : \mathrm{BX} = c_{\mathrm{A}} : c_{\mathrm{B}}$, the point S is independent of the choice of the point X, it is determined by c_{A} and c_{B} alone (more precisely, it is determined by the points A and B and the ratio $c_{\mathrm{A}} : c_{\mathrm{B}}$). The position of the point S on the line AB is given by $\mathrm{AS} = c \cdot c_{\mathrm{A}}/(c_{\mathrm{A}} - c_{\mathrm{B}})$, $\mathrm{BS} = c \cdot c_{\mathrm{B}}/(c_{\mathrm{A}} - c_{\mathrm{B}})$. The triangle RXS has the right angle at X, thus the point X lies on the circle with the diameter RS.

The converse is also true: if a triangle RXS has the right angle at X, then XR and XS are the bisectors of the internal resp. external angle at X (see Figure 3 — the point X is, as in Figure 2, above the line AB). Draw the parallel BU to RX and the parallel BV to SX, where the points U and V are on the line AX. We have $\mathrm{UX} : \mathrm{AX} = c_{\mathrm{B}} : c_{\mathrm{A}}$ and $\mathrm{VX} : \mathrm{AX} = \mathrm{BS} : \mathrm{AS} = c_{\mathrm{B}} : c_{\mathrm{A}}$, thus $\mathrm{UX} = \mathrm{VX}$. Since the triangle UBV has the right angle at B, it follows that $\mathrm{UX} = \mathrm{BX} = \mathrm{VX}$, whence XR and XS are indeed the bisectors of the internal resp. external angle at X of the triangle ABX.

For a point X not on the line AB the following are equivalent: (a) XR bisects the angle $\angle\mathrm{AXB}$, (b) $\mathrm{AX} : \mathrm{BX} = c_{\mathrm{A}} : c_{\mathrm{B}}$. We know that (a) implies (b). Suppose (b); then the bisector of the angle $\angle\mathrm{AXB}$ intersects the line segment AB in the point Y such that $\mathrm{AY} : \mathrm{BY} = \mathrm{AX} : \mathrm{BX} = c_{\mathrm{A}} : c_{\mathrm{B}}$, which means that $\mathrm{Y} = \mathrm{R}$. If a point X is on the line AB, then $\mathrm{AX} : \mathrm{BX} = c_{\mathrm{A}} : c_{\mathrm{B}}$ if and only if $\mathrm{X} = \mathrm{R}$ or $\mathrm{X} = \mathrm{S}$; in this special case the triangle ABX is one of the two degenerate triangles ABR, ABS.

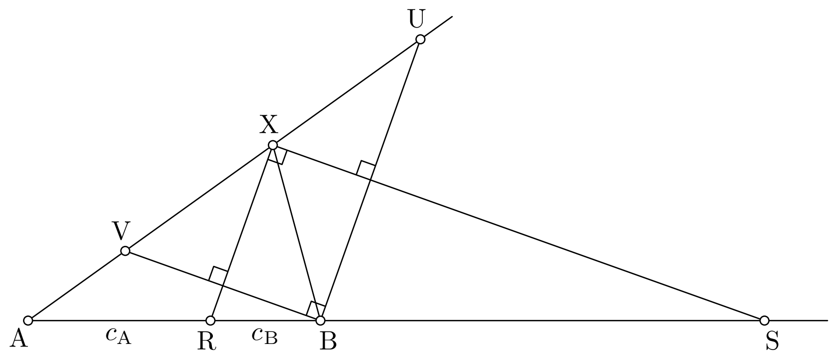


Figure 3

Let us sum up: the set of all points X in the plane such that AX : BX $= c_{\mathrm{A}} : c_{\mathrm{B}}$ is the circle with the diameter RS.[1]

The construction of the triangle ABC is shown in Figure 4: first we construct the

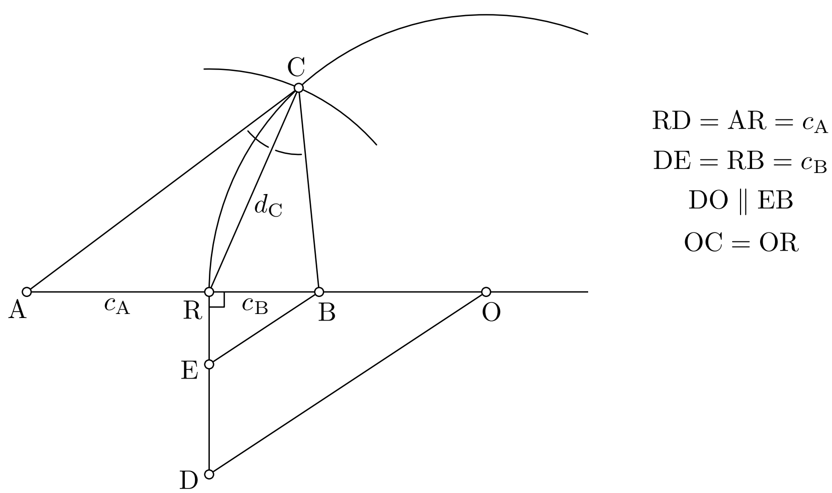


Figure 4

center O of the circle with the diameter RS, then we construct the point C as the vertex of the (isosceles) triangle ROC with the known sides OC = OR and RC $= d_{\mathrm{C}}$. The length of the diameter is $c_{\mathrm{B}} + c\cdot c_{\mathrm{B}}/(c_{\mathrm{A}} - c_{\mathrm{B}}) = 2c_{\mathrm{A}}\cdot c_{\mathrm{B}}/(c_{\mathrm{A}} - c_{\mathrm{B}})$; this is the basis of the construction of the center O, and besides that explains the constraint (4).

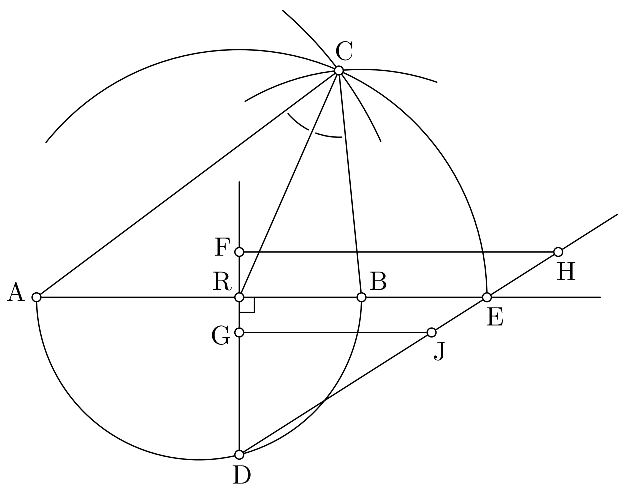
The construction we have just described has one blemish: when the ratio $c_{\mathrm{A}} : c_{\mathrm{B}}$ is approaching 1, the circle of Apollonius it employs is becoming larger and larger, until

[1]Given two distinct points A and B in the plane and a positive real number $k \neq 1$, the locus of all points X in the plane that have the constant ratio AX : BX $= k$ is known as the *circle of Apollonius* (of Perga). This circle is not so well-known (but should be) as the locus of all points Y in the plane subtending the constant angle $\angle\mathrm{AYB} = \alpha$; this is the reason why we have discussed the circle of Apollonius at length, giving full proofs.

at $c_{\mathrm{A}} : c_{\mathrm{B}} = 1$ the circle's center disappears into infinity and the circle straightens into a line, becoming the perpendicular bisector of the line segment AB. Let us see if we can read a decent enough construction from the formulas (2) and (3). Well, denoting by g the geometric mean of c_{A} and c_{B}, we can rewrite the formulas for a^2 and b^2 in the form

$$\frac{a}{c_{\mathrm{B}}} = \frac{\sqrt{g^2 + d_{\mathrm{C}}^2}}{g} = \frac{b}{c_{\mathrm{A}}}\,, \tag{5}$$

which suggests the construction in Figure 5. It would suffice to construct only one of



$$\mathrm{RA} = c_{\mathrm{A}}\,,\ \mathrm{RB} = c_{\mathrm{B}}$$
$$\big(\mathrm{RD} = g = \sqrt{c_{\mathrm{A}} c_{\mathrm{B}}}\big)$$
$$\mathrm{DF} = c_{\mathrm{A}}\,,\ \mathrm{DG} = c_{\mathrm{B}}$$
$$\mathrm{RE} = d_{\mathrm{C}}$$
$$\mathrm{FH} \parallel \mathrm{AB}\,,\ \mathrm{GJ} \parallel \mathrm{AB}$$
$$\mathrm{AC} = \mathrm{DH}\,,\ \mathrm{BC} = \mathrm{DJ}$$
$$\big(\mathrm{RC} = d_{\mathrm{C}}\big)$$

Figure 5

the sides a = BC, b = AC, since we know d_{C} = RC; nevertheless we construct both sides a and b, use them to find the vertex C, and then also draw the circle with center R and of radius d_{C}, which duly passes through the point C, thus confirming correctness of the construction. This construction does not explode when $c_{\mathrm{A}} : c_{\mathrm{B}}$ approaches 1.

Construct a triangle, given a, b, d_{C}

In this section we consider the following problem: construct a triangle ABC, given a, b, and d_{C} (Figure 6). Suppose we have the required triangle, and let CR be the bisector of its internal angle at C. Also, let CR$_{\mathrm{A}}$ be the bisector of the internal angle at C of the triangle ARC, and CR$_{\mathrm{B}}$ be the bisector of the internal angle at C of the triangle RBC (Figure 7). The three circles of Apollonius, with the diameters RS, R$_{\mathrm{A}}$S$_{\mathrm{A}}$, and R$_{\mathrm{B}}$S$_{\mathrm{B}}$, all pass through the vertex C. If we draw any two of these three circles, they intersect in the vertex C, and then for good measure we also draw the third circle, which must pass through the intersection point of the first two circles. Alas, we cannot construct any of the three circles, because we do not know the length of the side AB... However, if we start with any line segment A′B′, of an arbitrary length, we can construct a triangle A′B′C′, with the angle bisector C′R′, that is similar to the required triangle; for this

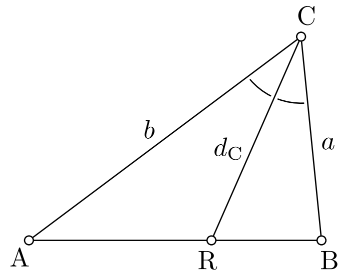


Figure 6

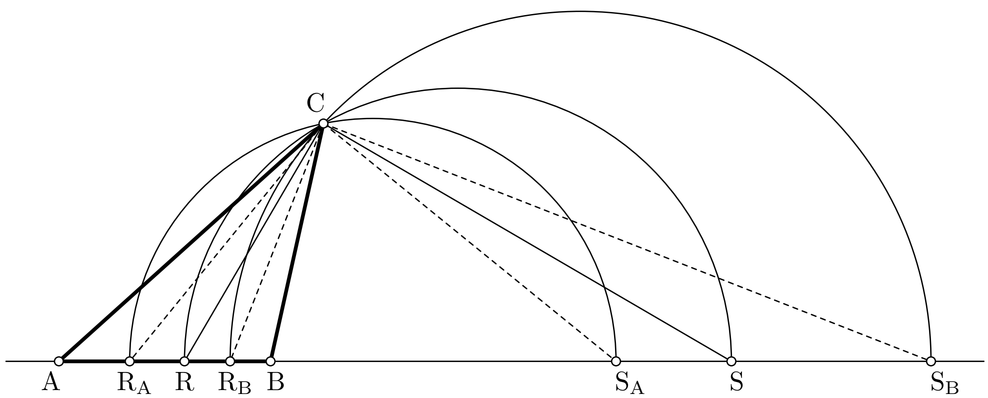


Figure 7

construction we need only the proportions $b : d_C : c$. Once we have the triangle A′B′C′, we rescale it by the ratio b : A′C′ (= a : B′C′ = d_C : R′C′), and we get our triangle ABC.

The construction we have just described is simple-minded, but simple it isn't. Besides that, there is the usual problem with the circles of Apollonius, which may be uncomfortably large when some of the proportions $a : b$, $a : d_C$, $d_C : b$ is close to 1. If the required triangle ABC exists, it is unique (up to congruence), but it does not always exist; the construction offers no clear criteria for which data it succeeds and for which it fails.

So, before we proceed to the next solution of the construction problem (the same one), let us determine the (necessary and sufficient) conditions for the existence of a solution. Assume that $a < b$; then $a+b > c > b-a$. Rewrite $d_C^2 = ab(a+b+c)(a+b-c)/(a+b)^2$ as

$$d_C^2 \;=\; ab\left(1 - \left(\frac{c}{a+b}\right)^2\right) . \tag{6}$$

We see that when c decreases from $a+b$ to $b-a$, then d_C^2 strictly increases from 0 to $4a^2b^2/(a+b)^2$, so d_C strictly increases from 0 to $2ab/(a+b)$, which is the harmonic mean of a and b. We conclude that the problem has a solution, a unique one, if and only if

$$0 \;<\; d_C \;<\; \frac{2ab}{a+b} . \tag{7}$$

There is a beautiful construction, one with an idea, of a triangle with given a, b, d_{C} at Cut The Knot ([1]). We start the construction by drawing three circles a, b, d with the center C and of respective radii a, b, d_{C} (Figure 8); we are assuming that $a < b$. We choose a point B on the circle a. By now we already have the vertices B and C of

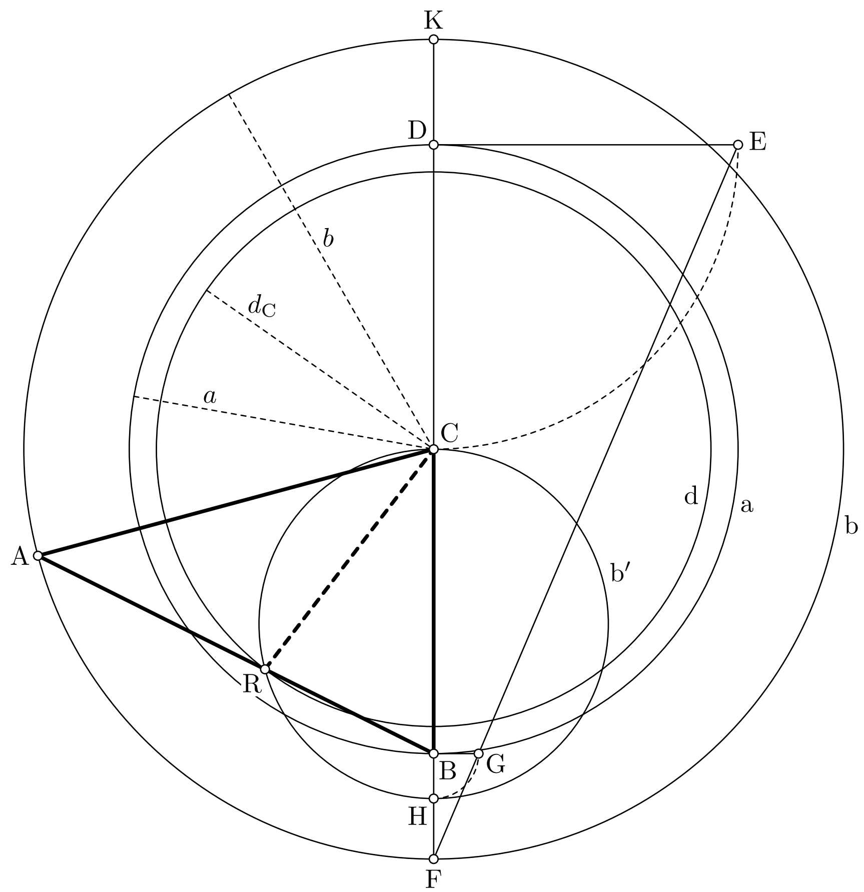


Figure 8. The construction at Cut The Knot of the triangle ABC, given a, b, d_{C}.

the triangle; it remains to find a point A on the circle b so that the triangle ABC will have the internal angle bisector at C of the prescribed length d_{C}. To this end we apply to the circle b the homothety χ with the center B and the ratio $a/(a+b)$, and obtain the circle b′. Let KF be the diameter of the circle b, as in Figure 8; then $\chi\mathrm{K} = \mathrm{C}$ and $\chi\mathrm{F} = \mathrm{H}$, where H is the point in the line segment BF that divides it in the proportion $\mathrm{BH} : \mathrm{HF} = a : b$. The diameter of the circle b′ is

$$\mathrm{CH} \;=\; a + \frac{a}{a+b} \cdot (b-a) \;=\; \frac{2ab}{a+b}\ ,$$

which comes as no great surprise to us: we need the circle b′ to intersect the circle d (just touching it is not good enough), which happens precisely when the given data satisfy the condition (7). Supposing that the condition is satisfied, let R be one of the two intersection points of the circles b′ and d, and let A be that of the two points in which the line BR intersects the circle b′ that is on the side of the point R opposite to the point B. Since clearly χA = R, we have BR : RA = $a : b$, which means that the line segment CR, of length d_{C}, is the bisector of the internal angle at the vertex C of the triangle ABC. Done.

Had we chosen the other intersection point R′ of the circles b′ and d, we would obtain a triangle A′BC congruent to the triangle ABC. (We know that the problem has only one solution, in case it exists.)

A remark about the construction [1] at Cut The Knot of a triangle, given a, b, l_c (the l_c is our d_{C}): the construction is correctly described, but the figure that accompanies it, here reproduced in Figure 9, is faulty. The proportions $a : b : l_c$ are faithfully

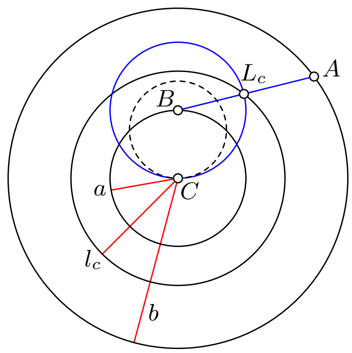


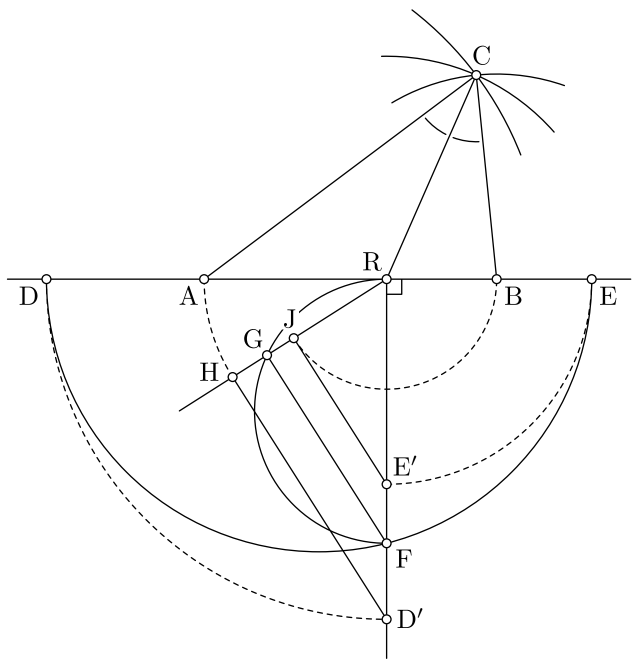
Figure 9. The figure accompanying the description of the triangle construction a, b, l_c at Cut The Knot.

reproduced: $a = 0.4\,b$, $l_c = 0.63\,b$. The blue circle, pasing through the point C, is said to be the image of the outer circle, the one with the center C of radius b, under the homothety with the center B and the ratio $a/(a+b)$. We see at a glance that something is wrong: the proportion $BL_c : L_cA$ (the point L_c is our R) should be $a : b = 0.4$, which it clearly isn't. The correct homothetic image of the outer circle is the dashed circle; since this circle does not intersect the circle with the center C of radius l_c, there is no solution. The prescribed l_c is too large, it is larger than the harmonic mean of a and b (the diameter of the dashed circle), which is approximately $0.57\,b$.

We will now give the third solution of the same problem, which will be another computed solution recast as a compass-and-straightedge construction. From (6) we get $c = (a+b)\sqrt{1 - d_{\mathrm{C}}^2/ab}$. Introducing $g := \sqrt{ab}$, we have

$$\frac{c_{\mathrm{A}}}{b} \;=\; \frac{\sqrt{g^2 - d_{\mathrm{C}}^2}}{g} \;=\; \frac{c_{\mathrm{B}}}{a}\;. \tag{8}$$

There is some symmetry between (8) and (5), yet the construction suggested by (8) (presented in Figure 10) is noticeably different from the construction suggested by (5)



$$\mathrm{RD} = b\,,\ \mathrm{RE} = a$$
$$\big(\mathrm{RF} = g = \sqrt{ab}\big)$$
$$\mathrm{FG} = d_{\mathrm{C}}$$
$$\mathrm{RD'} = b\,,\ \mathrm{RE'} = a$$
$$\mathrm{D'H} \parallel \mathrm{FG}\,,\ \mathrm{E'J} \parallel \mathrm{FG}$$
$$\mathrm{RA} = \mathrm{RH}\,,\ \mathrm{RB} = \mathrm{RJ}$$
$$\mathrm{AC} = b\,,\ \mathrm{BC} = a$$
$$\big(\mathrm{RC} = d_{\mathrm{C}}\big)$$

Figure 10

(Figure 5). The central step of the construction in Figure 10 is the finding of the point G on the semicircle with the diameter FR $= g$ such that FG $= d_{\mathrm{C}}$; with this step we in effect 'compute' the square root $\sqrt{g^2 - d_{\mathrm{C}}^2} = \mathrm{GR}$.

The foregoing construction does not reveal the correct upper bound on d_{C} which guarantees the existence of a solution: we obtain the point G whenever $d_{\mathrm{C}} < g = \sqrt{ab}$; however, the correct upper bound, the harmonic mean of a and b, is strictly smaller (when $a \neq b$) than their geometric mean.

References

[1] Alexander Bogomolny, "The many ways to construct a triangle III: a, b, l_c". Cut The Knot.