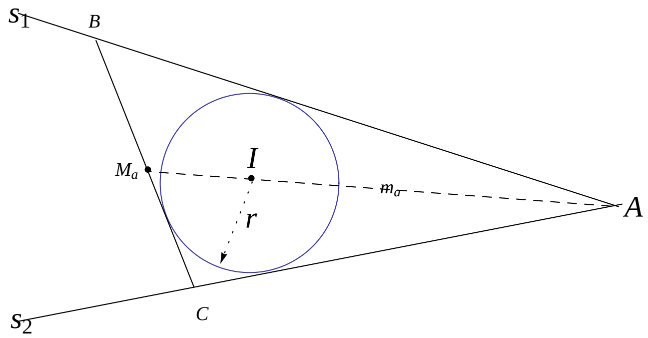
René Sperb , Dec. 1 , 2014

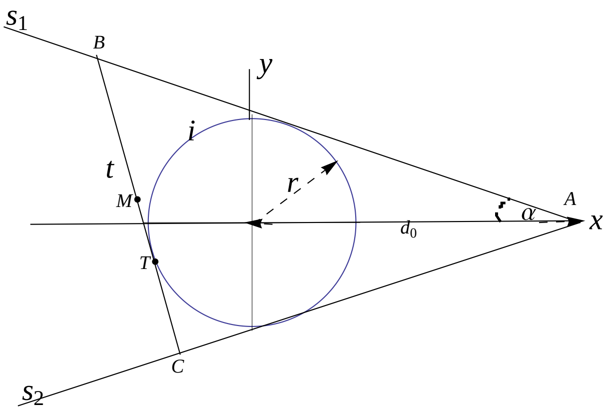
Triangle with given r , A , m_a



Analysis :

We have to find the points B , C such that the midpoint M_a has distance m_a from A.
To this end we let B and C move on the tangents s_1 and s_2 and determine the locus l of the corresponding midpoints M_a. Then we have to construct the point of intersection of this locus with the circle c_A of radius m_a around its center A.
It will be shown that l is a hyperbola with center I and asymptotes parallel to s_1 and s_2 .
The intersection of l and c_A can in this case be constructed with compass and straightedge .
We will see that this problem can be transformed into the problem of intersecting a circle and a straight line.
To do so there are some calculations in analytic geometry required.

(1) Determination of the locus l :



Setting α = A/2 and m = tan α the tangents to the incircle may be written as
y = ± m (x- d_0) and the tangent to i joining B and C as

x_1x + y_1y = r^2,

where (x_1, y_1) are the coordinates of T .
The coordiates of B (x_2, y_2) and C (x_3, y_3) are easily found from a linear system.
The coordinates of M are then

$\left(\frac{1}{2}(x_2 + x_3), \frac{1}{2}(y_2 + y_3)\right)$.

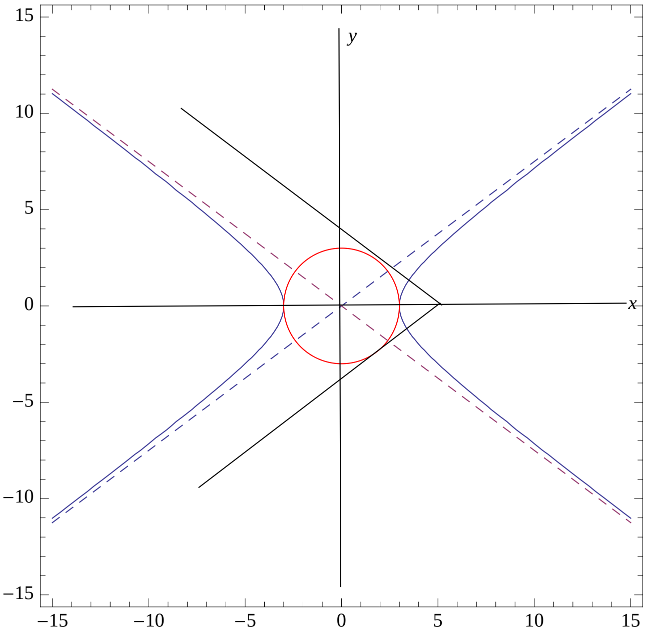
If one sets
x_1 = r cos t , y_1 = r sin t
one finds after routine calculation and some simplification that the coordinates of M are given by

$$x = \frac{r\,(d_0 + r\cos t)}{r + d_0\cos t}\;,\; y = \frac{r^2\sin t}{r + d_0\cos t}\;.$$

From this one can deduce that the locus of M is the hyperbola

$$\frac{x^2}{r^2} - \frac{y^2}{\frac{r^4}{d_0^{\,2} - r^2}} = 1.$$

The asymptotes are therefore parallel to the tangents s_1 , s_2 .
The situation is as shown in the next figure :



(2) Intersection of hyperbola and circle

Of course the general case cannot be reduced to a quadratic equation. But in the present case we have the circle

$(x - d_0)^2 + y^2 = m_a^2$

and the hyperbola

$\frac{x^2}{r^2} - \frac{y^2}{\frac{r^4}{d_2^{\,2}}} = 1$,

where $d_2 = \sqrt{d_0^{\,2} - r^2}$.

The intersection of the two curves requires the solution of the quadratic equation

(a) $x^2 \,(1+\left(\frac{r}{d_2}\right)^2) - 2 d_0 \mathrm{x} + d_0^{\,2} - m_a^{\,2} + \frac{r^4}{d_2^{\,2}} = 0$.

We now compare this quadratic equation with the equation we would get if we have to intersect the circle

$x^2 + y^2 = R^2$

and the line

y = m x + q .

This leads to the equation

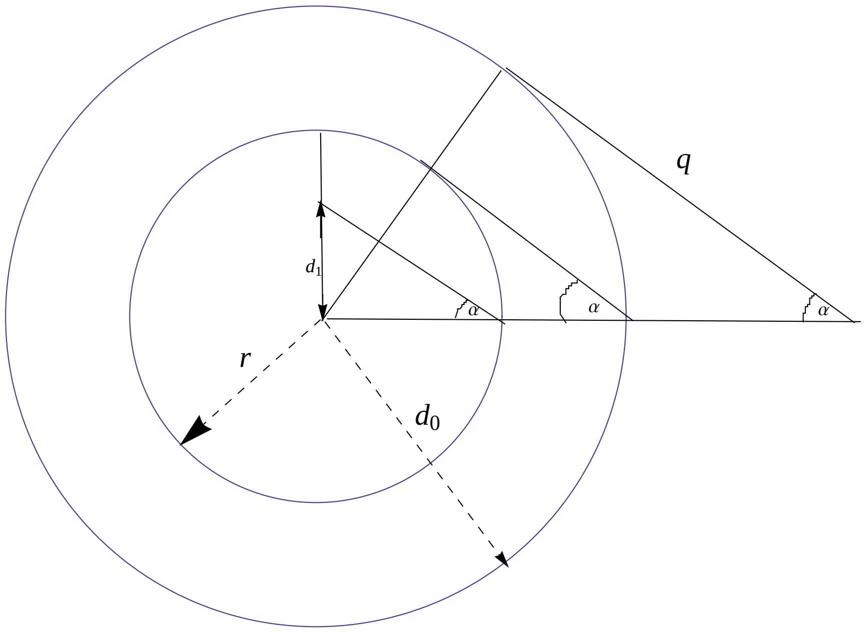
(b) $x^2\,(1+m^2)$ +2 m q x $+q^2 - R^2 = 0$

The comparison of (a) and (b) shows that if we select m = tan α , q = - $\frac{d_0}{\tan\alpha}$, d_1= r tan α , and

R= $\sqrt{q^2 + {d_1}^2 + {m_a}^2 - {d_0}^2}$,

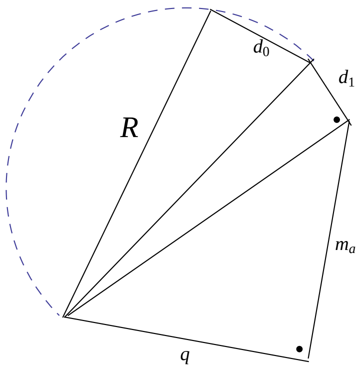
then the two equations (a) and (b) are equivalent.

The quantities d_0, d_1 , q can be read from the next figure

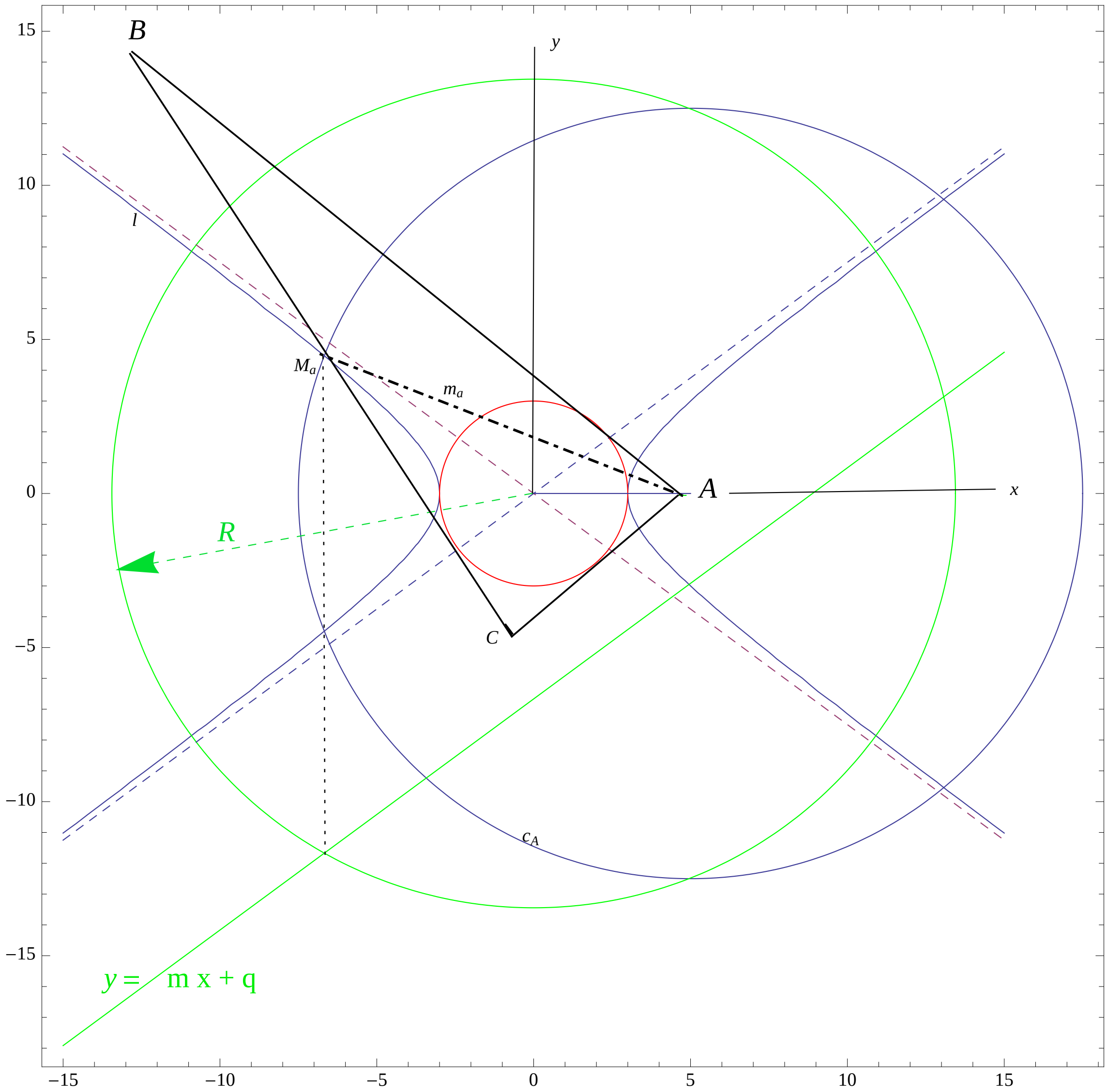


(3) Construction of R= $\sqrt{q^2 + {d_1}^2 + {m_a}^2 - {d_0}^2}$:

The principle of construction , based on Pythagoras's theorem is shown in the next figure:



The final construction with all the important elements is summarized in the next figure



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15
B
y
l
M_a
m_a
A
x
R
C
c_A
y = m x + q