Theorem 1.1

On each side of a parallelogram erect a regular quadrangle, lying exterior to the parallelogram. Then the mid segments connecting the neighboring apices of the regular quadrangles themselves are the apices of another regular quadrangle.

 Given a parallelogram ABCD. On its sides erect exterior regular quadrangles BEFC, CKPD, AMND, and MTQB.

Consider

O1 – the midpoint of QE;

O2– the midpoint of FK;

O3– the midpoint of PN;

O4– the midpoint of MT.

By turning around the point B at the angle of 90 ° counterclockwise vector $ \vec{BC}$ the vector becomes $\vec{BE}$, and the $\vec{BQ}$ in $\vec{BA}$. Therefore the vector $\vec{CQ}$ goes into the vector $\vec{EA}$, that means |$\vec{CQ}$|=|$\vec{EA}$|, $\vec{CQ}$=$\vec{EA}$ и ∠($\vec{CQ;}\vec{EA}$) = 90o.

 As H1 is the point of intersection of the diagonals of the quadrangle BEFC, H1 is the midpoint of EC and BF. Similarly, H2 is the midpoint of DK and CP, H3 is the midpoint of NA and MD, H4 is the midpoint of QA and TB.

As О1H1 is the mean line of ΔQEC, then O1H1= $\frac{1}{2}$CQ и O1H1||CQ

As O1H4- is the mean lineof ΔAQE, then O1H4= $\frac{1}{2} $EA и O1H4||EA

Thus, O1H1= O1H4и O1H1$⊥$ О1H4.

 Similarly, as H1O2 is the mean line of ΔBFK, then H1О2= $\frac{1}{2}$ BK и H1О2||BK

As ∠QBE=180°-∠ABC=180°- (180°-∠BCD) =∠BCD, то ∠QBC=∠BCK, therefore ΔBQC=ΔBCK (2 sides and the angle between them ), from the equality of thetriangles we get CQ=BK. As theopposite sides of the quadrangle QCKB are equal in pairs, then it is a parallelogram by idication.

As CQ||BK.

Then O1H1= H1О2 and O1H1|| H1О2. As O1H1 and H1О2 have a common point, they lie on one straight line.

Similarly, O1O4= 2O1H4 then points O1, O4, H4 lie on one straight line.

As O1H1=O1H4, then O1O4=O1O2 and O1O4$⊥$O1O2.

 Similarly we can prove for the other sides of the quadrangle, consequetly O1O2O3O4 is a square. Hence we obtain a proof for the theorem*.*

 The same task can be solved by the coordinate method.

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We introduce a rectangular coordinate system so as to coincide with the start point A, and the OX axis passes through the line BC. We find the coordinates of all the points in a given coordinate system

$$T\left(-a∙\sin(α);a∙\cos(α)\right);$$

$M\left(0;-b\right)$;

$B\left(a∙\cos(α);a∙\sin(α)\right)$;

$Q\left(a∙\cos(α)-a∙\sin(α);a∙\cos(α)+a∙\sin(α)\right)$;

$E\left(a∙\cos(α);a∙\sin(α)+b\right)$;

$$F\left(a∙\cos(α+b);a∙\sin(α)+b\right);$$

$K\left(a∙\cos(α+b)+a∙\sin(α);a∙\sin(α)-a∙\cos(α)\right)$;

$P\left(a∙\sin(α)+b;-a∙\cos(α)\right)$;

$N\left(b;-b\right)$;

$A\left(0;0\right)$;

$$O\_{1}\left(\frac{2∙a∙\cos(α)-a∙\sin(α)}{2};\frac{2∙a∙\sin(α)+a∙\cos(α)+b}{2}\right)$$

$$O\_{2}(\frac{2∙a∙\cos(α)+2b+a∙\sin(α)}{2};\frac{2∙a∙\sin(α)-a∙\cos(α)+b}{2})$$

$$O\_{3}(\frac{2b+a∙\sin(α)}{2};-\frac{a∙\cos(α)+b}{2})$$

$$O\_{4}(-\frac{a∙\sin(α)}{2};\frac{a∙\cos(α)-b}{2})$$

$$\vec{O\_{1}O\_{4}}(-a∙\cos(α);- a∙\sin(α)-b)$$

$$\vec{O\_{1}O\_{2}}(b+a∙\sin(α);-a∙\cos(α))$$

$|\vec{O\_{1}O\_{4}}|$=$\sqrt{(-a∙\cos(α))^{2}+(-a∙\sin(α)-b)^{2}}$

$$\left|\vec{O\_{1}O\_{2}}\right|=\sqrt{(b+a∙\sin(α))^{2}+(-a∙\cos(α))^{2}}$$

$$We get \left|\vec{O\_{1}O\_{4}}\right|=\left|\vec{O\_{1}O\_{2}}\right|$$

The scalar product of the vectors $\vec{O\_{1}O\_{2}}∙\vec{O\_{1}O\_{4}}$ = 0, then the angle between them is 90 °.

Similarly we can prove for the other sides of the quadrangle, consequetly O1O2O3O4 is a square by definition. Hence we obtain a proof for the theorem