1st proof: (This is the origin of Stathis Koutras’ Theorem)

Let $O$ be the intersection point of two different lines $(a),(b)$ and let $M_a, M_b, N_a, N_b$ be the orthogonal projections of segment $MN$ on $(a),(b)$ respectively. Let $A,B$ be points on $(a),(b)$ (on the same side of the rays), then prove:

\[
\frac{M_a N_a}{M_b N_b} = \frac{OB}{OA} \iff MN \perp AB
\]

\[
\text{Proof}
\]

- **Proof**
- Let \( \frac{M_a N_a}{M_b N_b} = \frac{OB}{OA} \): (1). If $OS \parallel MN$ and let $K,L$ be the orthogonal projections of $S$ on $(a),(b)$ respectively. Then

\[
OK = M_a N_a, OL = M_b N_b \quad \text{(orthogonal projections of equal and parallel segments on the same line or parallel lines are equal)} \quad \text{and} \quad OA' = OL = M_b N_b, OB' = OK = M_a N_a \quad \text{(3) with $A',B'$ on the same sides of rays with initial point $O$.}
\]

Obviously

\[
\triangle OB'A' = \triangle OKL \Rightarrow \angle B'A'O = \angle OLK = \angle OSK \Rightarrow SO \perp A'B' \quad \text{(4)}
\]

- But \( \frac{OB'}{OA'} = \frac{M_a N_a}{M_b N_b} = \frac{OB}{OA} \quad \text{Thales Th.} \Rightarrow AB \parallel A'B' \Rightarrow MN \perp AB
\]

- Let $MN \perp AB \Rightarrow OS \perp AB$ and let $T = OS \cap AB$, Then the quadrilaterals $ATKS, BTLS$ are cyclic on circles of the same chord $TS$ therefore $OA \cdot OK = OS \cdot OT = OB \cdot OL \Rightarrow \frac{OK}{OL} = \frac{OB}{OA} \Rightarrow \frac{M_a N_a}{M_b N_b} = \frac{OB}{OA}.

Brussels 6 January 2017
Stathis Koutras
2\textsuperscript{nd} proof: Let \( O \) be the intersection point of two different lines \((a),(b)\) and let \( M_a M_b N_a N_b \) be the orthogonal projections of segment \( MN \) on \((a),(b)\) respectively. Let \( A,B \) be points on \((a),(b)\) (on the same side of the rays), then prove: \[
\frac{M_N N_b}{M_a N_a} = \frac{OB}{OA} \iff MN \perp AB
\]

\begin{proof}

\begin{itemize}
\item Let \[
\frac{M_N N_b}{M_a N_a} = \frac{OB}{OA}; (1)
\]
Let \( OS \parallel MN \) and let \( K,L \) be the orthogonal projections of \( S \) on \((a),(b)\) respectively.
Then \[
OK = M_a N_a \cdot OL = M_b N_b; (2)
\]
(orthogonal projections of equal and parallel segments on the same line or parallel lines are equal). Let \( A' = SL \cap (a), B' = SK \cap (b) \).
Then obviously \( S \) is the orthocenter of triangle \( \triangle OA'B' \). \[
\Rightarrow OS \perp A'B'; (3)
\]

From the concyclic points \( K,A',B',L \) \( \Rightarrow OK \cdot OA' = OL \cdot OB' \Rightarrow \)
\[
\frac{OK}{OL} = \frac{OB'}{OA'} \Rightarrow \frac{OB}{OA} = \frac{OB'}{OA'} \Rightarrow AB \parallel A'B' \Rightarrow MN \perp AB
\]

\item Let \( MN \perp AB \Rightarrow OS \perp AB \) and because \( OS \perp A'B' \Rightarrow AB \parallel A'B' \) \[
\Rightarrow \frac{OB}{OA} = \frac{OB'}{OA'} = \frac{OK}{OL} = \frac{M_a N_a}{M_b N_b}
\]
\end{itemize}
\end{proof}

\textbf{Comments:} Translated by Takis Chronopoulos (06 January 2017), The first proof has been published before online, here in Greek [http://mathematica.gr/forum/viewtopic.php?p=181515#p181515](http://mathematica.gr/forum/viewtopic.php?p=181515#p181515) (08 August 2013), The second proof is a new one.