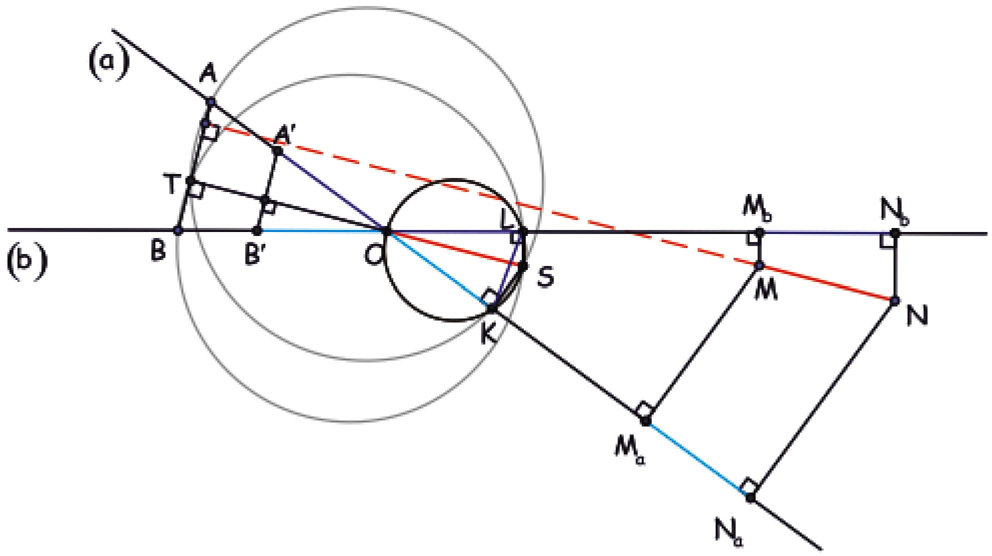
1^{st} proof: (This is the origin of Stathis Koutras' Theorem)

Let O be the intersection point of two different lines $(a),(b)$ and let M_aM_b, N_aN_b be the orthogonal projections of segment MN on $(a),(b)$ respectively. Let A,B be points on $(a),(b)$ (on the same side of the rays) , then prove:

$$\frac{M_aN_a}{M_bN_b}=\frac{OB}{OA}\Leftrightarrow MN\perp AB$$



<u>Proof</u>

• Let $\boxed{\dfrac{M_aN_a}{M_bN_b}=\dfrac{OB}{OA}}:(1)$. If $OS \overset{=}{\parallel} MN$ and let K,L be the orthogonal projections of S on $(a),(b)$ respectively. Then $\boxed{OK=M_aN_a, OL=M_bN_b}:(2)$ (orthogonal projections of equal and parallel segments on the same line or parallel lines are equal) and $\boxed{OA'=OL=M_bN_b, OB'=OK=M_aN_a}:(3)$ with A',B' on the same sides of rays with initial point O.

Obviously

$$\triangle OB'A' \overset{(SAS)}{=} \triangle OKL \Rightarrow \angle B'A'O=\angle OLK \overset{O,L,S,K\text{ concyclic}}{=} \angle OSK \overset{SK\perp(a)}{\Rightarrow} \boxed{SO\perp A'B'}:(4)$$

But $\dfrac{OB'}{OA'}\overset{(3)}{=}\dfrac{M_aN_a}{M_bN_b}\overset{(1)}{=}\dfrac{OB}{OA}\overset{\text{Thales Th.}}{\Rightarrow} AB\parallel A'B' \overset{(4)}{\Rightarrow}\boxed{MN\perp AB}$

• Let $MN\perp AB\Rightarrow OS\perp AB$ and let $T\equiv OS\cap AB$, Then the quadrilaterals $ATKS, BTLS$ are cyclic on circles of the same chord TS therefore $OA\cdot OK=OS\cdot OT=OB\cdot OL\Rightarrow \dfrac{OK}{OL}=\dfrac{OB}{OA}\overset{(2)}{\Rightarrow}\boxed{\dfrac{M_aN_a}{M_bN_b}=\dfrac{OB}{OA}}$.

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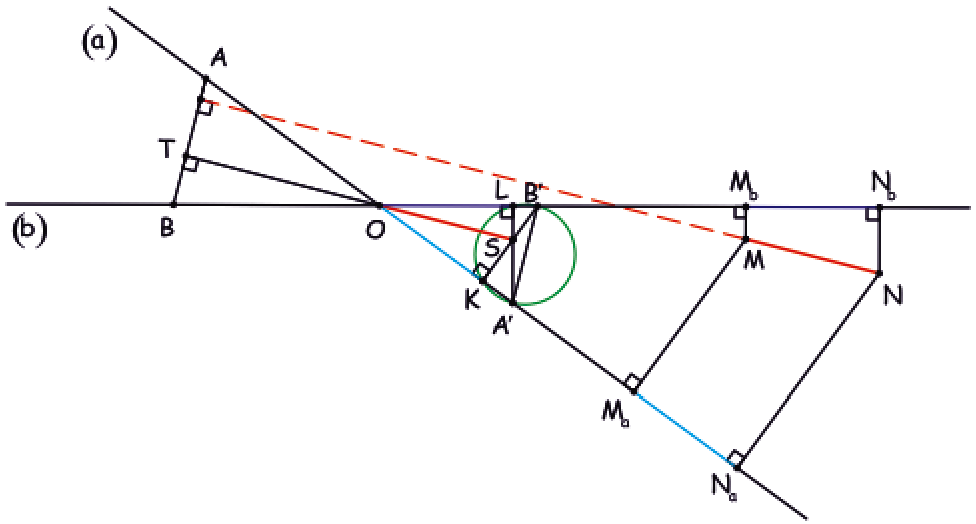
2nd proof: Let O be the intersection point of two different lines $(a),(b)$ and let M_aM_b, N_aN_b be the orthogonal projections of segment MN on $(a),(b)$ respectively. Let A,B be points on $(a),(b)$ (on the same side of the rays), then prove: $\frac{M_aN_a}{M_bN_b}=\frac{OB}{OA}\Leftrightarrow MN\perp AB$



<u>Proof</u>

• Let $\boxed{\frac{M_aN_a}{M_bN_b}=\frac{OB}{OA}}:(1)$. Let $OS =\parallel MN$ and let K,L be the orthogonal projections of S on $(a),(b)$ respectively. Then $\boxed{OK=M_aN_a, OL=M_bN_b}:(2)$ (orthogonal projections of equal and parallel segments on the same line or parallel lines are equal). Let $A'\equiv SL\cap(a), B'\equiv SK\cap(b)$. Then obviously S is the orthocenter of triangle $\triangle OA'B' \Rightarrow \boxed{OS\perp A'B'}:(3)$

From the concyclic points $K,A',B',L \Rightarrow OK\cdot OA'=OL\cdot OB' \Rightarrow \frac{OK}{OL}=\frac{OB'}{OA'}\overset{(1),(2)}{\Rightarrow}\frac{OB}{OA}=\frac{OB'}{OA'}\overset{\text{Thales Th. converse}}{\Rightarrow} AB\parallel A'B' \overset{(3),OS\parallel MN}{\Rightarrow}\boxed{MN\perp AB}$

• Let $MN\perp AB \overset{OS\parallel MN}{\Rightarrow} OS\perp AB$ and because $OS\perp A'B'\Rightarrow AB\parallel A'B' \overset{\text{Thales Th.}}{\Rightarrow}\frac{OB}{OA}=\frac{OB'}{OA'}\overset{K,L,B',A'\text{ concyclic}}{=}\frac{OK}{OL}\overset{(2)}{=}\frac{M_aN_a}{M_bN_b}$

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<u>Comments:</u> Translated by Takis Chronopoulos (06 January 2017), The first proof has been published before online, here in Greek http://mathematica.gr/forum/viewtopic.php?p=181515#p181515 (08 August 2013), The second proof is a new one.