# On a class of three-variable inequalities

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## 1 Theorem

Let $a, b, c$ be real numbers satisfying $a+b+c=1$. By the AM - GM inequality, we have $ab+bc+ca \le \frac{1}{3}$, therefore setting $ab+bc+ca = \frac{1-q^2}{3}$ $(q \ge 0)$, we will find the maximum and minimum values of $abc$ in terms of $q$.
If $q = 0$, then $a = b = c = \frac{1}{3}$, therefore $abc = \frac{1}{27}$. If $q \ne 0$, then $(a-b)^2+(b-c)^2+(c-a)^2 > 0$. Consider the function $f(x) = (x-a)(x-b)(x-c) = x^3 - x^2 + \frac{1-q^2}{3}x - abc$. We have

$$f'(x) = 3x^2 - 2x + \frac{1-q^2}{3}$$

whose zeros are $x_1 = \dfrac{1+q}{3}$, and $x_2 = \dfrac{1-q}{3}$.
We can see that $f'(x) < 0$ for $x_2 < x < x_1$ and $f'(x) > 0$ for $x < x_2$ or $x > x_1$. Furthermore, $f(x)$ has three zeros: $a, b$, and $c$. Then

$$f\left(\frac{1-q}{3}\right) = \frac{(1-q)^2(1+2q)}{27} - abc \ge 0$$

and

$$f\left(\frac{1+q}{3}\right) = \frac{(1+q)^2(1-2q)}{27} - abc \le 0.$$

Hence

$$\frac{(1+q)^2(1-2q)}{27} \le abc \le \frac{(1-q)^2(1+2q)}{27}$$

and we obtain

**Theorem 1.1** *If $a, b, c$ are arbitrary real numbers such that $a+b+c=1$, then setting $ab+bc+ca = \frac{1-q^2}{3}$ $(q \ge 0)$, the following inequality holds*

$$\frac{(1+q)^2(1-2q)}{27} \le abc \le \frac{(1-q)^2(1+2q)}{27}.$$

Or, more general,

**Theorem 1.2** *If $a, b, c$ are arbitrary real numbers such that $a+b+c=p$, then setting $ab+bc+ca = \frac{p^2-q^2}{3}$ $(q \ge 0)$ and $r = abc$, we have*

$$\frac{(p+q)^2(p-2q)}{27} \le r \le \frac{(p-q)^2(p+2q)}{27}.$$

This is a powerful tool since the equality holds if and only if $(a-b)(b-c)(c-a) = 0$.

Here are some identities which we can use with this theorem

$$a^2+b^2+c^2=\frac{p^2+2q^2}{3}$$
$$a^3+b^3+c^3=pq^2+3r$$
$$ab(a+b)+bc(b+c)+ca(c+a)=\frac{p(p^2-q^2)}{3}-3r$$
$$(a+b)(b+c)(c+a)=\frac{p(p^2-q^2)}{3}-r$$
$$a^2b^2+b^2c^2+c^2a^2=\frac{(p^2-q^2)^2}{9}-2pr$$
$$ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2)=\frac{(p^2+2q^2)(p^2-q^2)}{9}-pr$$
$$a^4+b^4+c^4=\frac{-p^4+8p^2q^2+2q^4}{9}+4pr$$

## 2 Applications

**2.1** *Let $a, b, c$ be positive real numbers such that $a+b+c=1$. Prove that*

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+48(ab+bc+ca)\geq 25.$$

***Solution.*** We can easily check that $q\in[0,1]$, by using the theorem we have

$$LHS=\frac{1-q^2}{3r}+16(1-q^2)\geq\frac{9(1+q)}{(1-q)(1+2q)}+16(1-q^2)=\frac{2q^2(4q-1)^2}{(1-q)(1+2q)}+25\geq 25.$$

The inequality is proved. Equality holds if and only if $a=b=c=\frac{1}{3}$ or $a=\frac{1}{2}, b=c=\frac{1}{4}$ and their permutations.

**2.2** [Vietnam 2002] *Let $a, b, c$ be real numbers such that $a^2+b^2+c^2=9$. Prove that*

$$2(a+b+c)-abc\leq 10.$$

***Solution.*** The condition can be rewritten as $p^2+2q^2=9$. Using our theorem, we have

$$LHS=2p-r\leq 2p-\frac{(p+q)^2(p-2q)}{27}=\frac{p(5q^2+27)+2q^3}{27}.$$

We need to prove that

$$p(5q^2+27)\leq 270-2q^3.$$

This follows from

$$(270-2q^3)^2\geq p^2(5q^2+27)^2,$$

or, equivalently,

$$27(q-3)^2(2q^4+12q^3+49q^2+146q+219)\geq 0.$$

The inequality is proved. Equality holds if and only if $a=b=2, c=-1$ and their permutations.

**2.3** [Vo Quoc Ba Can] *For all positive real numbers $a, b, c$, we have*

$$\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}+11\sqrt{\frac{ab+bc+ca}{a^2+b^2+c^2}}\geq 17.$$

***Solution.*** Because the inequality is homogeneous, without loss of generality, we may assume that $p = 1$. Then $q \in [0, 1]$ and the inequality can be rewritten as

$$\frac{1-q^2}{3r} + 11\sqrt{\frac{1-q^2}{1+2q^2}} \geq 20.$$

Using our theorem, it suffices to prove

$$11\sqrt{\frac{1-q^2}{1+2q^2}} \geq 20 - \frac{9(1+q)}{(1-q)(1+2q)} = \frac{-40q^2+11+11}{(1-q)(1+2q)}.$$

If $-40q^2 + 11q + 11 \leq 0$, or $q \geq \frac{11+3\sqrt{209}}{80}$, it is trivial. If $q \leq \frac{11+3\sqrt{209}}{80} < \frac{2}{3}$, we have

$$\frac{121(1-q^2)}{(1+2q^2)} - \frac{(-40q^2+11q+11)^2}{(1-q)^2(1+2q)^2} = \frac{3q^2(11-110q+255q^2+748q^3-1228q^4)}{(1+2q^2)(1-q)^2(1+2q)^2}.$$

On the other hand,

$$\begin{aligned}11 - 110q + 255q^2 + 748q^3 - 1228q^4 &= q^4\left(\frac{11}{q^4} - \frac{110}{q^3} + \frac{255}{q^2} + \frac{748}{q} - 1228\right)\\ &\geq q^4\left(\frac{11}{(2/3)^4} - \frac{110}{(2/3)^3} + \frac{255}{(2/3)^2} + \frac{748}{2/3} - 1228\right) = \frac{2435}{16}q^4 \geq 0.\end{aligned}$$

The inequality is proved. Equality occurs if and only if $a = b = c$.

**2.4** [Vietnam TST 1996] *Prove that for any $a, b, c \in \mathbb{R}$, the following inequality holds*

$$(a+b)^4 + (b+c)^4 + (c+a)^4 \geq \frac{4}{7}(a^4+b^4+c^4).$$

***Solution.*** If $p = 0$ the inequality is trivial, so we will consider the case $p \neq 0$. Without loss of generality, we may assume $p = 1$. The inequality becomes

$$3q^4 + 4q^2 + 10 - 108r \geq 0$$

Using our theorem, we have

$$3q^4 + 4q^2 + 10 - 108r \geq 3q^4 + 4q^2 + 10 - 4(1-q)^2(1+2q) = q^2(q-4)^2 + 2q^4 + 6 \geq 0.$$

The inequality is proved. Equality holds only for $a = b = c = 0$.

**2.5** [Pham Huu Duc, MR1/2007] *Prove that for any positive real numbers $a, b,$ and $c$,*

$$\sqrt{\frac{b+c}{a}} + \sqrt{\frac{c+a}{b}} + \sqrt{\frac{a+b}{c}} \geq \sqrt{6 \cdot \frac{a+b+c}{\sqrt[3]{abc}}}$$

***Solution.*** By Holder's inequality, we have

$$\left(\sum_{\text{cyc}} \sqrt{\frac{b+c}{a}}\right)^2 \left(\sum_{\text{cyc}} \frac{1}{a^2(b+c)}\right) \geq \left(\sum_{\text{cyc}} \frac{1}{a}\right)^3$$

It suffices to prove that

$$\left(\sum_{\text{cyc}} \frac{1}{a}\right)^3 \geq \frac{6(a+b+c)}{\sqrt[3]{abc}} \sum_{\text{cyc}} \frac{1}{a^2(b+c)}$$

Setting $x = \frac{1}{a}, y = \frac{1}{b}, z = \frac{1}{c}$, the inequality becomes

$$(x+y+z)^3 \geq 6\sqrt[3]{xyz}(xy+yz+zx)\sum_{\text{cyc}} \frac{x}{y+z},$$

or

$$(x+y+z)^3 \geq \frac{6\sqrt[3]{xyz}(xy+yz+zx)((x+y+z)^3 - 2(x+y+z)(xy+yz+zx) + 3xyz)}{(x+y)(y+z)(z+x)}.$$

By the AM - GM inequality,

$$(x+y)(y+z)(z+x) = (x+y+z)(xy+yz+zx) - xyz \geq \frac{8}{9}(x+y+z)(xy+yz+zx).$$

It remains to prove that

$$4(x+y+z)^4 \geq 27\sqrt[3]{xyz}((x+y+z)^3 - 2(x+y+z)(xy+yz+zx) + 3xyz).$$

Setting $p = x+y+z, xy+yz+zx = \frac{p^2-q^2}{3}$ $(p \geq q \geq 0)$, the inequality becomes

$$4p^4 \geq 9\sqrt[3]{xyz}(p^3 + 2pq^2 + 9xyz).$$

Applying our theorem, it suffices to prove that

$$4p^4 \geq 9\sqrt[3]{\frac{(p-q)^2(p+2q)}{27}}\left(p^3 + 2pq^2 + \frac{(p-q)^2(p+2q)}{3}\right),$$

$$4p^4 \geq \sqrt[3]{(p-q)^2(p+2q)}(3p^3 + 6pq^2 + (p-q)^2(p+2q)).$$

Setting $u = \sqrt[3]{\frac{p-q}{p+2q}} \leq 1$, the inequality is equivalent to

$$4(2u^3+1)^4 \geq 27u^2(4u^9 + 5u^6 + 2u^3 + 1),$$

or

$$f(u) = \frac{(2u^3+1)^4}{u^2(4u^9+5u^6+2u^3+1)} \geq \frac{27}{4}$$

We have

$$f'(u) = \frac{2(2u^3+1)^3(u^3-1)(2u^3-1)(2u^6+2u^3-1)}{u^3(u^3+1)^2(4u^6+u^3+1)^2}$$

$$f'(u) = 0 \Leftrightarrow u = \sqrt[3]{\frac{\sqrt{3}-1}{2}}, \text{ or } u = \frac{1}{\sqrt[3]{3}}, \text{ or } u = 1.$$

Now, we can easily verify that

$$f(u) \geq \min\left\{f\left(\sqrt[3]{\frac{\sqrt{3}-1}{2}}\right), f(1)\right\} = \frac{27}{4},$$

which is true. The inequality is proved. Equality holds if and only if $a = b = c$.

**2.6** [Darij Grinberg] *If $a, b, c \geq 0$, then*

$$a^2 + b^2 + c^2 + 2abc + 1 \geq 2(ab + bc + ca).$$

***Solution.*** Rewrite the inequality as

$$6r + 3 + 4q^2 - p^2 \geq 0.$$

If $2q \geq p$, it is trivial. If $p \geq 2q$, using the theorem, it suffices to prove that

$$\frac{2(p-2q)(p+q)^2}{9} + 3 + 4q^2 - p^2 \geq 0,$$

or

$$(p-3)^2(2p+3) \geq 2q^2(2q+3p-18).$$

If $2p \leq 9$, we have $2q + 3p \leq 4p \leq 18$, therefore the inequality is true. If $2p \geq 9$, we have

$$2q^2(2q+3p-18) \leq 4q^2(2p-9) \leq p^2(2p-9) = (p-3)^2(2p+3) - 27 < (p-3)^2(2p+3).$$

The inequality is proved. Equality holds if and only if $a = b = c = 1$.

**2.7** [Schur's inequality] *For any nonnegative real numbers $a, b, c$,*

$$a^3 + b^3 + c^3 + 3abc \geq ab(a+b) + bc(b+c) + ca(c+a).$$

***Solution.*** Because the inequality is homogeneous, we can assume that $a+b+c = 1$. Then $q \in [0,1]$ and the inequality is equivalent to

$$27r + 4q^2 - 1 \geq 0.$$

If $q \geq \frac{1}{2}$, it is trivial. If $q \leq \frac{1}{2}$, by the theorem we need to prove that

$$(1+q)^2(1-2q) + 4q^2 - 1 \geq 0,$$

or

$$q^2(1-2q) \geq 0,$$

which is true. Equality holds if and only if $a = b = c$ or $a = b, c = 0$ and their permutations.

**2.8** [Pham Kim Hung] *Find the greatest constant $k$ such that the following inequality holds for any positive real numbers $a, b, c$*

$$\frac{a^3+b^3+c^3}{(a+b)(b+c)(c+a)} + \frac{k(ab+bc+ca)}{(a+b+c)^2} \geq \frac{3}{8} + \frac{k}{3}.$$

***Solution.*** For $a = b = 1 + \sqrt{3}$ and $c = 1$, we obtain $k \leq \frac{9\left(3+2\sqrt{3}\right)}{8} = k_0$. We will prove that this is the desired value. Let $k_0$ be a constant satisfying the given inequality. Without loss of generality, assume that $p = 1$. Then $q \in [0,1]$ and the inequality becomes

$$\frac{3(3r+q^2)}{-3r+1-q^2} + \frac{k_0(1-q^2)}{3} \geq \frac{3}{8} + \frac{k_0}{3}.$$

It is not difficult to verify that this is an increasing function in terms of $r$. If $2q \geq 1$, we have

$$VT \geq \frac{3q^2}{1-q^2} + \frac{k_0(1-q^2)}{3} \geq 1 + \frac{k_0}{4} \geq \frac{3}{8} + \frac{k_0}{3}.$$

(since this is an increasing function in terms of $q^2 \geq \frac{1}{4}$)

If $2q \leq 1$, using our theorem, it suffices to prove that

$$\frac{3((1+q)^2(1-2q)+9q^2)}{-(1+q)^2(1-2q)+9(1-q^2)} + \frac{k_0(1-q^2)}{3} \geq \frac{3}{8} + \frac{k_0}{3}.$$

We have

$$LHS - RHS = \frac{3q^2\left(3+2\sqrt{3}\right)\left(2\sqrt{3}-1-q\right)\left(q-2+\sqrt{3}\right)^2}{8(q+1)(q-2)^2} \geq 0.$$

The inequality is proved, and we conclude that $k_{\max} = k_0$.

**2.9** [Pham Huu Duc] *For all positive real numbers* $a, b$ *and* $c$,

$$\frac{1}{a^2+bc}+\frac{1}{b^2+ca}+\frac{1}{c^2+ab} \le \frac{(a+b+c)^2}{3(ab+bc+ca)}\left(\frac{1}{a^2+b^2}+\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2}\right).$$

***Solution.*** Because the inequality is homogeneous, we may assume that $p=1$. Then $q \in [0,1]$ and by the AM - GM and Schur's inequalities, we have $\frac{(1-q^2)^2}{9} \ge 3r \ge \max\left\{0, \frac{1-4q^2}{9}\right\}$. After expanding, we can rewrite the given inequality as

$$\begin{aligned} f(r) = &-486(9-q^2)r^3+27(q^6+64q^4-35q^2+24)r^2+9(4q^2-1)(11q^4-4q^2+2)r \\ &+q^2(1-q^2)^3(2q^4+8q^2-1) \ge 0. \end{aligned}$$

We have

$$\begin{aligned} f'(r) &= 9(-162(9-q^2)r^2+6(q^6+64q^4-35q^2+24)r+(4q^2-1)(11q^4-4q^2+2)) \\ f''(r) &= 54(-54(9-q^2)r+q^6+64q^4-35q^2+24) \\ &\ge 54(-2(1-q^2)^2(9-q^2)+q^6+64q^4-35q^2+24) = 162(q^6+14q^4+q^2+2) > 0. \end{aligned}$$

Hence $f'(r)$ is an increasing function.
Now, if $1 \le 2q$, then

$$f'(r) \ge f'(0) = (4q^2-1)(11q^4-4q^2+2) \ge 0.$$

If $1 \ge 2q$, then

$$f'(r) \ge f'\left(\frac{1-4q^2}{27}\right) = (1-4q^2)(q^2+2)(2q^4+17q^2+6) \ge 0.$$

In any case, $f(r)$ is an increasing function.
If $1 \le 2q$, then $f(r) \ge f(0) = q^2(1-q^2)^3(2q^4+8q^2-1) \ge 0$, and we are done. If $1 \ge 2q$, using our theorem, we have

$$f(r) \ge f\left(\frac{(1+q)^2(1-2q)}{27}\right) = \frac{1}{81}q^2(2-q)(q+1)^2(6q^3+4q^2-7q+4)(5q^2-2q+2)^2 \ge 0.$$

The proof is complete. Equality holds if and only if $a=b=c$.

**2.10** [Nguyen Anh Tuan] *Let* $x, y, z$ *be positive real numbers such that* $xy+yz+zx+xyz=4$. *Prove that*

$$\frac{x+y+z}{xy+yz+zx} \le 1+\frac{1}{48}\cdot((x-y)^2+(y-z)^2+(z-x)^2).$$

***Solution.*** Since $x, y, z > 0$ and $xy+yz+zx+xyz=4$, there exist $a, b, c > 0$ such that $x=\frac{2a}{b+c}, y=\frac{2b}{c+a}, z=\frac{2c}{a+b}$. The inequality becomes

$$P(a,b,c) = \frac{(a+b+c)^2\sum_{\text{cyc}}(a^2-b^2)^2}{(a+b)^2(b+c)^2(c+a)^2} - \frac{6\sum_{\text{cyc}}a(a+b)(a+c)}{\sum_{\text{cyc}}ab(a+b)}+12 \ge 0.$$

Because the inequality is homogeneous we can assume that $p=1$. Then $q \in [0,1]$ and after some computations, we can rewrite the inequality as

$$f(r) = 729r^3+27(22q^2-1)r^2+27(6q^4-4q^2+1)r+(q^2-1)(13q^4-5q^2+1) \le 0.$$

We have

$$f'(r) = 27(r(81r+44q^2-2)+6q^4-4q^2+1).$$

By Schur's inequality,

$$81r+44q^2-2 \geq 3(1-4q^2)+44q^2-2 = 1+32q^2 > 0.$$

Hence $f'(r) \geq 0$, and $f(r)$ is an increasing function. Then by our theorem we have

$$f(r) \leq f\left(\frac{(1-q)^2(1+2q)}{27}\right) = \frac{2}{27}q^2(q-1)(q+2)^2(4q^4+14q^3+15q^2-7q+1) \leq 0.$$

The inequality is proved. Equality holds if and only if $x=y=z$.

**2.11** [Nguyen Anh Tuan] *For all nonnegative real numbers $a, b, c$*

$$\sqrt{(a^2-ab+b^2)(b^2-bc+c^2)}+\sqrt{(b^2-bc+c^2)(c^2-ca+a^2)}+\sqrt{(c^2-ca+a^2)(a^2-ab+b^2)} \geq a^2+b^2+c^2.$$

***Solution.*** After squaring both sides, we can rewrite the inequality as

$$2\sqrt{\prod_{\text{cyc}}(a^2-ab+b^2)}\left(\sum_{\text{cyc}}\sqrt{a^2-ab+b^2}\right) \geq \left(\sum_{\text{cyc}} ab\right)\left(\sum_{\text{cyc}} a^2\right) - \sum_{\text{cyc}} a^2b^2.$$

By the AM - GM inequality,

$$\sqrt{a^2-ab+b^2} \geq \frac{1}{2}\cdot(a+b),\ \sqrt{b^2-bc+c^2} \geq \frac{1}{2}\cdot(b+c), \sqrt{c^2-ca+a^2} \geq \frac{1}{2}\cdot(c+a).$$

It suffices to prove that

$$2\sqrt{\prod_{\text{cyc}}(a^2-ab+b^2)}\left(\sum_{\text{cyc}} a\right) \geq \left(\sum_{\text{cyc}} ab\right)\left(\sum_{\text{cyc}} a^2\right) - \sum_{\text{cyc}} a^2b^2.$$

Because this inequality is homogeneous, we can assume $p=1$. Then $q \in [0,1]$ and the inequality is equivalent to

$$2\sqrt{-72r^2+3(1-10q^2)r+q^2(1-q^2)^2} \geq 6r+q^2(1-q^2),$$

or

$$f(r) = 324r^2-12r(q^4-11q^2+1)-q^2(4-q^2)(1-q^2)^2 \leq 0.$$

It is not difficult to verify that $f(r)$ is a convex function, then using our theorem, we have

$$f(r) \leq \max\left\{f(0), f\left(\frac{(1-q)^2(1+2q)}{27}\right)\right\}.$$

Furthermore,

$$f(0) = -q^2(4-q^2)(1-q^2)^2 \leq 0,$$

$$f\left(\frac{(1-q)^2(1+2q)}{27}\right) = \frac{1}{9}q^2(q-1)^3(q+2)(9q^2+q+2) \leq 0.$$

Our proof is complete. Equality holds if and only if $a=b=c$ or
$a=t \geq 0$, $b=c=0$, and their permutations.

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