

ASYMPTOTIC BEHAVIOR OF CERTAIN DUCCI SEQUENCES

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ABSTRACT. The Ducci map is considered applied to vectors in \mathbb{R}^n . It is shown that for certain starting vectors of odd length, the corresponding sequence of iterates asymptotically approaches a periodic vector, but never becomes periodic itself. On the other hand, for any starting vectors not of this type, it is shown that the corresponding Ducci sequence becomes eventually periodic. A simple method for determining if a given vector is of this certain type is presented. In this way, the asymptotic behavior of all Ducci sequences of vectors in \mathbb{R}^n , n odd, is characterized. The map is then applied to the specific case of \mathbb{R}^3 . It is shown that in this case, the Ducci sequences of vectors of the aforementioned type approach the zero vector only.

1. INTRODUCTION

Consider the map $f : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ given by

$$f(a, b, c, d) = (|a - b|, |b - c|, |c - d|, |d - a|),$$

and for a fixed starting vector, form the sequence $\{f^i(a, b, c, d)\}_{i=0}^{\infty}$ in \mathbb{R}^4 . It was a question about the limiting behavior of such sequences which Ciamberlini and Marengoni [16] attributed to E. Ducci in 1937. Since then, such sequences have most commonly come to be known as *Ducci sequences* or the *four-number game*.

The literature devoted to this topic has become quite extensive. This seems in part to be due to the ease with which the question can be posed, particularly in the case of vectors in \mathbb{Z}^4 . For example, by experimenting with a variety of different vectors $(a, b, c, d) \in \mathbb{Z}^4$, one will likely find that the corresponding Ducci sequence reaches the vector $(0, 0, 0, 0)$ in a relatively small number of steps. The question of whether or not this must always occur has appeared in a number of books on mathematics meant for general audiences [28, 29, 45, 46, 51], as well as at various times in the mathematics literature [1, 24, 27, 47].

That the zero vector will indeed always be reached after a finite number of iterations of f applied to vectors in \mathbb{Z}^4 is confirmed in [28]. And in [47], a more general question also appears. For any integer $n \geq 2$, define the map $f : \mathbb{Z}^n \rightarrow \mathbb{Z}^n$ by making the obvious generalization to get

$$f(x_1, x_2, \dots, x_n) = (|x_1 - x_2|, |x_2 - x_3|, \dots, |x_n - x_1|).$$

The question then becomes, for any starting vector (x_1, x_2, \dots, x_n) , does the corresponding Ducci sequence contain the zero vector after only a finite number of iterations? The original proof that this is in fact the case if and only if n is a power of 2 appeared in the seminal paper of Ciamberlini and Marengoni [16], and was followed by a surprising variety of later proofs using various techniques [2, 10, 12, 20, 22, 26, 40, 53].

One way in which these results have been refined has been to consider the number of iterations needed to reach the zero vector when n is a power of 2. In [49, 53] bounds are given for the number of such iterations, and refining this somewhat in the case of $n = 4$, [4, 19, 46] also show that for any $M \in \mathbb{N}$, there exist integer vectors which take at least M iterations to reach $(0, 0, 0, 0)$. Even more specifically, [5] uses the tribonacci numbers to show how to find a starting vector in \mathbb{Z}^4 which takes any specified number of iterations to reach the zero vector, with [42] giving similar results. Many of these results are generalized to arbitrary powers of 2 in [10, 12, 22].

When n is not a power of 2 and integer entries are considered, the sequence of iterates of f , although not necessarily convergent to zero, still displays an eventual regularity. In particular, for any integer $n \geq 2$, if $(x_1, x_2, \dots, x_n) \in \mathbb{Z}^n$, then the corresponding Ducci sequence will eventually contain a vector which is a scalar multiple of a vector in $\{0, 1\}^n$ [21, 22, 23, 43]. It is clear that under application of f any such 0-1 vector can only give rise to another 0-1 vector. Hence, the subsequent sequence of iterates must then cycle, and so in this sense all integer Ducci sequences have tails which consist of repeating segments. Of course, in the case that n is a power of 2 these segments contain only the zero vector. Also in analogy with the power of 2 case, for any $n \geq 2$ it has been shown that there exist initial vectors which can take any specified number of iterations to reach one of these repeating

segments [6, 17, 18, 22, 33, 34]. Others have taken up questions related to these repeating segments, such as their lengths [7, 10, 11, 21, 25, 36], while yet others have generalized to consider the Ducci map over more general abelian groups [8]. Rather than restricting attention to vectors with integer entries, in this paper we will be interested in the Ducci map applied to vectors with real-number entries.

Perhaps somewhat surprisingly, the greatest amount of work on the Ducci map applied to vectors with real entries has occurred for $n = 4$. In fact, Lotan [32] proved that the situation here differs from the integer-entry case in that there exist vectors in \mathbb{R}^4 for which the corresponding Ducci sequence never reaches the zero vector. In particular, by letting q represent the positive solution to the cubic equation $x^3 - x^2 - x - 1 = 0$, the vector $(1, q, q^2, q^3)$ will correspond to a Ducci sequence which never reaches $(0, 0, 0, 0)$. Much more than this, [32] proved that every vector in \mathbb{R}^4 , except ones which can be obtained from $(1, q, q^2, q^3)$ through the obvious transformations (translating, shifting, reflecting, and scaling), will reach $(0, 0, 0, 0)$ in a finite number of steps. Therefore, modulo these exceptional vectors, of which there is essentially only one, the behavior of f on \mathbb{R}^4 is the same as on \mathbb{Z}^4 . But, although the vector $(1, q, q^2, q^3)$ does not reach the zero vector in a finite number of iterations, the corresponding Ducci sequence does converge to $(0, 0, 0, 0)$. Consequently, even in this exceptional case, the asymptotic behavior is consistent with the behavior of all other vectors. It is also interesting to note that this result has been, in whole or in part, proved and reproved a number of times since [3, 12, 30, 31, 37, 38, 46]. In [48] the probabilities of randomly choosing a vector in \mathbb{R}^4 which takes any specified number of iterations to reach the zero vector are calculated. Further mention of some of the previous work on this problem can be found in [14, 39].

A variety of generalizations of the Ducci map problem have also been considered. One such example is that of a Ducci process [52]. Here, rather than the map which takes (x_1, x_2, \dots, x_n) to $(|x_1 - x_2|, \dots, |x_n - x_1|)$, the function $(x, y) \mapsto |x - y|$ is replaced by a more general map $\phi(x, y)$. Then the Ducci process is such that (x_1, x_2, \dots, x_n) maps to the vector $(\phi(x_1, x_2), \dots, \phi(x_n, x_1))$. Such generalized mappings were further considered

by [35, 44]. Other generalizations of the original Ducci map utilize weightings. For example, the original mapping utilizes the weighting $(1, -1)$, whereas one other possibility which has recently been considered [13, 15] utilizes the weighting $(-1, 2, -1)$ to define the map on vectors in \mathbb{R}^3 where $(x_1, x_2, x_3) \mapsto (|2x_1 - x_3 - x_2|, |2x_2 - x_1 - x_3|, |2x_3 - x_2 - x_1|)$.

With all the work done on the topic of Ducci sequences, one is led to inquire as to their applications. The study of Ducci sequences falls under the subject of difference equations, which in general are of great interest and application. According to the Journal of Difference Equations and Applications website, difference equations can have applications to “non-linear dynamics, chaos theory, complex dynamics, mathematical biology, discrete control theory, oscillation theory, Symmetries and integrable systems, functional equations, special functions and orthogonal polynomials, numerical analysis, combinatorics, computational linear algebra, and dynamic equations on times scales.” While currently there may not be any direct application of Ducci sequences, it is altogether possible that one day a practical use for them will be found.

In addition to this, another important reason for studying Ducci sequences is out of interest. The topic of Ducci sequences is a playful one, as evidenced by its being called the “four-number game,” as mentioned above. The Ducci map is a deceptively simple function, and through better understanding of its workings, we can better begin to understand the limits of our own intellect. The Ducci map presents many puzzles that are interesting to solve; we will examine a few of these in the upcoming paper.

For the present work, attention will be focused on the Ducci map f applied to vectors in \mathbb{R}^n , n odd. We begin by characterizing the asymptotic behavior of vectors in these spaces, and then we proceed to completely describe end behavior in the case $n = 3$. We will conclude with a description of the difficulty in generalizing some of our results for the case $n = 3$.

Why $n = 3$? Much work has been done on the \mathbb{R}^4 case, but in contrast there is very little that has appeared which explicitly treats real-valued entries for $n = 3$. One exception is [50], where the question of which such sequences result in cycles is considered. Another exception is [41]. There, among other things, it was shown that if n is a power of 2, then any vector $v \in \mathbb{R}^n$ will give rise to a Ducci sequence for which $\lim\{|f^n(v)|\} = 0$. Then, in [9], this was

further generalized to show that for any positive integer $n \geq 2$ and any vector $v \in \mathbb{R}^n$, the sequence $\{f^n(v)\}$ converges to a periodic vector. In the case that n is a power of 2, [41] tells us that this periodic vector is the trivial one, namely $(0, 0, \dots, 0)$, whereas otherwise it will be a vector which is a scalar multiple of an element in $\{0, 1\}^n$.

Although [9] sheds some light on the behavior of the Ducci map for $n = 3$, it still leaves open certain questions regarding the behavior in this case. For example, [9] constructs a vector in \mathbb{R}^7 which corresponds to a sequence which asymptotically approaches a nontrivial periodic vector, but which has none of its entries themselves periodic. We will show that this behavior cannot occur for $n = 3$. That is, either a vector in \mathbb{R}^3 gives rise to a sequence which is eventually periodic, or it asymptotically approaches the trivial periodic vector $(0, 0, 0)$. Furthermore, we apply our general result mentioned above to show which vectors in \mathbb{R}^3 exhibit each of these separate behaviors.

2. HETEROGENEOUS VECTORS IN \mathbb{R}^n

We begin with a definition that will greatly simplify notation.

Definition 2.1. Suppose that $x \in \mathbb{R}$, $n \in \mathbb{Z}^+$. Then $(x)_n$ represents the vector (x, x, \dots, x) in \mathbb{R}^n .

We consider the Ducci map f defined in general on \mathbb{R}^n . As shown in [21, 22, 23, 43], for any vector $v \in \mathbb{Z}^n$, there exists $i \in \mathbb{N}$ such that $f^i(v)$ is a scalar multiple of a periodic vector in $\{0, 1\}^n$. Clearly this result also holds for vectors with rational entries. In fact, given $v \in \mathbb{R}^n$, $v = (v_1, v_2, \dots, v_n)$, if there exist $\alpha, x \in \mathbb{R}$, $\alpha \neq 0$, such that $\alpha(v + (x)_n) \in \mathbb{Q}^n$, then since $f(\alpha(v + (x)_n)) = |\alpha|(|v_1 - v_2|, |v_2 - v_3|, \dots, |v_n - v_1|) = |\alpha|f(v)$, we see that any such vector will also correspond to a Ducci sequence $\{f^i(v)\}$ which will eventually be periodic. Of course, there are many vectors which do not have this property, such as $(0, 1, \sqrt{2})$ in \mathbb{R}^3 and $(0, 1, 2, 3, \sqrt{2})$ in \mathbb{R}^5 . Therefore, to analyze the behavior of f , we are motivated to examine this latter type of vector.

Definition 2.2. We will call a vector $v \in \mathbb{R}^n$ *heterogeneous* if for all $x, \alpha \in \mathbb{R}$, $\alpha \neq 0$, we have $\alpha(v + (x)_n) \notin \mathbb{Q}^n$. Furthermore, a vector's *heterogeneity* is its state of being either heterogeneous or not heterogeneous.

As it turns out, given a vector v , it is easier to determine whether or not it is heterogeneous than the definition may make it appear. The following lemmas provide a simplified method for determining heterogeneity.

Lemma 2.3. *Let $v = (v_1, v_2, \dots, v_n)$. Then v is heterogeneous if and only if for every scalar $k \neq 0$, $k(v + (-v_1)_n) \notin \mathbb{Q}^n$.*

Proof. The left-to-right case is clear. We proceed to prove the right-to-left case by proving its contrapositive. Suppose that v is not heterogeneous. Then there exists real numbers x, k , $k \neq 0$, such that $k(v + (x)_n) \in \mathbb{Q}^n$. That is, $(kv_1 + kx, kv_2 + kx, \dots, kv_n + kx) = (q_1, q_2, \dots, q_n)$ where the $q_i \in \mathbb{Q}$. So $kv_i - kv_1 = q_i - q_1 \in \mathbb{Q}$, and $k(v + (-v_1)_n) \in \mathbb{Q}^n$ as desired. \square

It is clear from the definition that if we wish to determine the heterogeneity of a given arbitrary starting vector (v_1, v_2, \dots, v_n) , we can equivalently determine the heterogeneity of $(0, v_2 - v_1, \dots, v_n - v_1)$. The above result greatly simplifies determination of heterogeneity for vectors of the latter type (since the first entry of such vectors is 0). Vectors of this form also have several other useful results associated with them, as we shall soon see. For the moment, we thus restrict attention to vectors of this type.

Lemma 2.4. *Let $v = (0, v_2, \dots, v_n) \in \mathbb{R}^n$ have at least one nonzero entry. Let $x \in \mathbb{N}$ be the least natural number such that $v_x \neq 0$. Then v is heterogeneous if and only if $v_i/v_x \notin \mathbb{Q}$ for some $x < i \leq n$.*

Proof. We again proceed by contrapositive. For the left-to-right case, suppose that for every $x < i \leq n$, $v_i/v_x \in \mathbb{Q}$. Then $(1/v_x)v = (0, \dots, 0, 1, v_{x+1}/v_x, \dots, v_n/v_x) \in \mathbb{Q}^n$ (where there are $x - 1$ preceding zeros), and v is not heterogeneous, as desired. For the right-to-left case, suppose that v is not heterogeneous. Then by Lemma 2.3 there exists a nonzero scalar k such that $kv = (0, \dots, 0, v_x k, \dots, v_n k) \in \mathbb{Q}^n$. So then for any $x < i \leq n$, $(v_i k)/(v_x k) = v_i/v_x \in \mathbb{Q}$, completing the proof by contrapositive. \square

Corollary 2.5. *Let $v = (0, v_2, \dots, v_n) \in \mathbb{R}^n$ have at least one nonzero entry. Then v is heterogeneous if and only if $v_i/v_j \notin \mathbb{Q}$, $v_j \neq 0$, for some $2 \leq i, j \leq n$.*

Proof. Let $x \in \mathbb{N}$ be the least natural number such that $v_x \neq 0$. For the left-to-right case, suppose that v is heterogeneous. Then by Lemma 2.4, there exists an i such that v_i/v_x is irrational, as desired. For the right to left case, suppose that there exist i, j such that $v_i/v_j, v_j \neq 0$, is irrational. Then $\frac{v_i/v_x}{v_j/v_x}$ is also irrational. Hence at least one of $v_i/v_x, v_j/v_x$ is irrational, and by Lemma 2.4 v is heterogeneous. \square

We now have the tools to tackle the following theorem. As we shall see, this theorem, coupled with the obvious fact that if v is not heterogeneous, neither is $f(v)$, shows that the set of heterogeneous vectors of odd length is closed under f .

Theorem 2.6. *Let $w = (w_1, w_2, \dots, w_n) \in \mathbb{R}^n$ be a heterogeneous vector of odd length. Then $f(w)$ is also heterogeneous.*

Proof. Let the smallest entry of w be w_k . Then let $v = (0, v_2, \dots, v_n) = (0, w_{k+1} - w_k, \dots, w_n - w_k, w_1 - w_k, \dots, w_{k-1} - w_k)$ be the vector obtained upon rotating $w - (w_k)_n$ to the left $k - 1$ times. This rotation will not affect the entries of the subsequent images under repeated application of f , but merely which entry is read first. Hence, we see that w and v must have the same heterogeneity. Note that since w_k is the smallest of w 's entries, all v_i are positive.

It is thus sufficient for us to show that if v is heterogeneous, then $f(v)$ is also heterogeneous. As before, we proceed by a proof by contrapositive. Suppose $f(v)$ is not heterogeneous. We calculate $f(v) - (v_2)_n = (0, |v_3 - v_2| - v_2, \dots, |v_n| - v_2) = (0, v'_2, v'_3, \dots, v'_n) = v'$.

Case 1: Each v'_i is rational. We define the sequence $L_i, 2 \leq i < n$, as

$$L_i = \begin{cases} -1, & \text{if } v_i < v_{i+1}; \\ 1, & \text{if } v_i \geq v_{i+1}. \end{cases}$$

We see that $L_i v'_i = L_i(|v_{i+1} - v_i| - v_2) = v_i - v_{i+1} - L_i v_2$ for $2 \leq i \leq n - 1$. Then consider the sum

$$L_2 v'_2 + L_3 v'_3 + \dots + L_{n-1} v'_{n-1} + v'_n$$

This equals

$$v_2 - v_3 - L_2v_2 + v_3 - v_4 - L_3v_2 + \dots + v_{n-1} - v_n - L_{n-1}v_1 + v_n - v_2.$$

We rearrange the above to

$$v_2 - v_3 + v_3 - v_4 + \dots + v_{n-1} - v_n + v_n - v_2 - v_2(L_2 + L_3 + \dots + L_{n-1}),$$

which becomes

$$-(L_2 + L_3 + \dots + L_{n-1})v_2.$$

But since each $L_i v'_i$ (and v'_i) is rational, it follows that their sum must also be rational. Since each L_i is either 1 or -1, and there are an odd number of them in the above sum, it follows that $L_2 + L_3 + \dots + L_{n-1}$ is nonzero. Hence v_2 is rational as well. Since $v'_2 = |v_3 - v_2| - v_2$ is rational, it follows that v_3 is rational as well. Similarly, $v'_3 = |v_4 - v_3| - v_2$ is rational, so v_4 must be rational. In this way, we see that each v_i , $2 \leq i \leq n$ must be rational. Hence v is not heterogeneous, and this case is complete.

Case 2: There exists a v'_k that is irrational, where v' and each v'_i is defined as above. Then consider the vector $Q = (1/v'_k)v$. Note that Q is such that $f(Q) = |1/v'_k|f(v)$, so $f(Q)$ is not heterogeneous. Let $Q' = f(Q) - (v_2)_n/|v'_k| = \frac{v'}{|v'_k|}$. Since Q' is not heterogeneous, by Corollary 2.5, for all integers $2 \leq x, y \leq n$ the ratio of the x and y th entries of Q' are rational (provided the y th entry is nonzero). Since the k th entry of Q' is rational (specifically, its value is either 1 or -1), it follows that every entry of Q' is rational. So we apply Case 1 to see that Q is not heterogeneous. Hence $(v'_k)Q = v$ is not heterogeneous, as desired. \square

The following result follows from inductively utilizing Theorem 2.6.

Corollary 2.7. *Let $w = (w_1, w_2, \dots, w_n) \in \mathbb{R}^n$ be a heterogeneous vector of odd length. Then for any $i \in \mathbb{N}$, $f^i(w)$ is also heterogeneous.*

We have now reached a point where these results can be combined into a cohesive whole. Suppose we wish to analyze the asymptotic behavior of an arbitrary vector $w = (w_1, w_2, \dots, w_n)$ in \mathbb{R}^n , n odd. If all of the w_i are equal, then it is clear that $f(w) = (0)_n$ is the trivial periodic vector. If not all the w_i are equal, we can rotate this vector to obtain $v = (v_1, v_2, \dots, v_n)$ with

$v_1 \neq v_2$, which has the same heterogeneity as w . From here, we may consider the vector $v' = (v - (v_1)_n)/(v_2 - v_1) = (0, 1, \dots, \frac{v_n}{v_2 - v_1})$, which must have the same heterogeneity as v and therefore also as w . By Lemma 2.4, v' is heterogeneous if and only if it has at least one irrational entry. In this manner, we can easily determine the heterogeneity of w .

After determining the heterogeneity of w , the result of the above corollary becomes very useful in describing the asymptotic behavior of w 's corresponding Ducci sequence. If w is not heterogeneous, the corresponding Ducci sequence will eventually contain a periodic vector that is a scalar multiple of an element of $\{0, 1\}^n$, as stated at the beginning of this section. On the other hand, if w is heterogeneous, then by Corollary 2.7 the Ducci sequence $\{f^i(w)\}$ will contain only heterogeneous vectors. Therefore, it will never contain a multiple of a vector in $\{0, 1\}^n$, all of which are clearly not heterogeneous. But since those vectors which are in a cycle are multiples of vectors in $\{0, 1\}^n$ this result ultimately implies that heterogeneous vectors cannot give rise to Ducci sequences which reach a vector which is in a cycle. Hence by [9], we see that the corresponding Ducci sequence of such a vector will converge asymptotically to, but never contain, a vector that is a scalar multiple of an element of $\{0, 1\}^n$.

3. DUCCI SEQUENCES IN \mathbb{R}^3

We now shift our attention to the specific case of $n = 3$. We will show that in this case, all heterogeneous vectors converge to the zero vector. As mentioned previously, this stands in contrast to $n = 7$, in which case [9] has provided a vector that converges to a nontrivial periodic vector. At the end of the paper, we will explain the difficulty in generalizing our result here to higher n .

Before we proceed, note that a rearrangement of the entries in a vector prior to application of f will not alter the magnitudes of the entries in the image, only their relative positions within the vector, and will also have no effect on the magnitudes of the entries in subsequent images. Heterogeneity is similarly unaffected. Hence, we can reorder the entries of any vector however we like without altering the properties we are studying.

We now define two functions which will allow us to shift our analysis of the Ducci mapping f to an analysis of a mapping defined on a subset of the plane \mathbb{R}^2 . Specifically, first define g on the set of heterogeneous vectors v in \mathbb{R}^3 which have one entry zero and two positive (necessarily distinct) entries to those points in the plane $\{(a, b) : a/b \notin \mathbb{Q}, 0 < a < b\}$ by $g(v) = (a, b)$, where b is the largest of v 's elements and a is the second largest of v 's elements. Considering the same subset of \mathbb{R}^2 , next define the function

$$h : \{(a, b) : a/b \notin \mathbb{Q}, 0 < a < b\} \rightarrow \{(a, b) : a/b \notin \mathbb{Q}, 0 < a < b\}$$

by

$$h(a, b) = \begin{cases} (2a - b, a), & \text{if } b < 2a; \\ (b - 2a, b - a), & \text{if } 2a < b. \end{cases}$$

The function h is the analog of the Ducci map f restricted to heterogeneous vectors, a statement we make more precise in the following.

Lemma 3.1. *Suppose v is a heterogeneous vector in \mathbb{R}^3 . Then there exist unique $k, K \in \mathbb{R}$ such that $g(f(v) - (K)_3) = h(g(v - (k)_3))$. In particular, k equals the smallest entry of v and K the smallest entry of $f(v)$.*

Proof. Suppose $v = (a, b, c)$, where, without a loss of generality we assume $a < b < c$. As $f(v) = f(v + (j)_3)$ for any real j , we let $j = -a$ to obtain $v + (j)_3 = (0, b - a, c - a) = (0, \alpha, \beta)$, where $\alpha = b - a$ and $\beta = c - a$, so that $f(0, \alpha, \beta) = f(v)$. Thus, our result will follow if we can find unique k and K such that $g(f(0, \alpha, \beta) - (K)_3) = h(g(v - (k)_3))$.

Calculating, we see that $f(0, \alpha, \beta) = (\alpha, \beta - \alpha, \beta)$. For $f(0, \alpha, \beta) - (K)_3$ to be in the domain of g , one of its entries must be zero. As β is the greatest of the entries of $f(0, \alpha, \beta)$, and is in fact strictly larger than α and $\beta - \alpha$ since v is heterogeneous, we have only two cases to consider.

Case 1: If $2\alpha > \beta$, and therefore $\alpha > \beta - \alpha$, then set $K = \beta - \alpha$, and $f(0, \alpha, \beta) - (K)_3 = (2\alpha - \beta, 0, \alpha)$. Note that $\alpha > 2\alpha - \beta$ since this is equivalent to $\beta > \alpha$. Also note that since $(0, \alpha, \beta)$ is heterogeneous, so too is $f(0, \alpha, \beta)$, and thus $f(0, \alpha, \beta) - (K)_3$ (which has one zero and two positive entries) is in the domain of g .

Case 2: If $2\alpha < \beta$, and therefore $\alpha < \beta - \alpha$, then set $K = \alpha$, and $f(0, \alpha, \beta) - (K)_3 = (0, \beta - 2\alpha, \beta - \alpha)$. Clearly $\beta - \alpha > \beta - 2\alpha$. The previous comment regarding the fact that $f(0, \alpha, \beta) - (K)_3$ is in the domain of g applies here as well.

Thus we have

$$g(f(v) - (K)_3) = \begin{cases} (2\alpha - \beta, \alpha), & \text{if } 2\alpha > \beta \\ (\beta - 2\alpha, \beta - \alpha), & \text{if } 2\alpha < \beta, \end{cases}$$

and upon substituting for α and β , we find that

$$g(f(v) - (K)_3) = \begin{cases} (2b - a - c, b - a), & \text{if } 2b > a + c \\ (c + a - 2b, c - b), & \text{if } 2b < a + c. \end{cases}$$

On the other hand, we have $g(v - (a)_3) = (b - a, c - a)$, and so by now calculating $h(g(v - (k)_3))$ for $k = a$, we reach the desired conclusion. The uniqueness of k and K follow from the obvious fact that given a vector $v \in \mathbb{R}^3$, for at most one value of j can $v - (j)_3$ be in the domain of g . \square

This lemma can be generalized via induction to apply to any iterate of f , as stated in the following lemma. It is then on the basis of Lemma 3.2 that we are able to analyze the action of h in order to determine the behavior of f on heterogeneous vectors.

Lemma 3.2. *Let $i \in \mathbb{N}$ and suppose v is a heterogeneous vector in \mathbb{R}^3 . Then there exist unique real numbers k, K such that*

$$g(f^i(v) - (K)_3) = h^i(g(v - (k)_3)).$$

Here k is the smallest entry of v and K is the smallest entry of $f^i(v)$.

4. ASYMPTOTIC BEHAVIOR OF HETEROGENEOUS VECTORS IN \mathbb{R}^3

We now focus our analysis on those points $(a, b) \in \mathbb{R}^2$ such that $0 < a < b$ and $a/b \notin \mathbb{Q}$. To this end, the next lemma demonstrates that any such point with $b < 2a$ will eventually be shifted under iteration by h to the region where $b > 2a$.

Lemma 4.1. *Let (a, b) be in the domain of h and such that $b < 2a$. Then there exists $i \in \mathbb{N}$ such that $h^i(a, b) = (\alpha, \beta)$ satisfies $\beta > 2\alpha$.*

Proof. Letting $(a_0, b_0) = (a, b)$, suppose, on the contrary, that for each $i \geq 0$, $h^i(a, b) = (a_i, b_i)$ satisfies $b_i < 2a_i$. Then, the mapping h takes (a_i, b_i) to $(2a_i - b_i, a_i)$ for all positive i , and we conclude, by induction, that $h^i(a, b) = ((i+1)a - ib, ia - (i-1)b)$ for all $i \geq 1$. This implies that for all $i \in \mathbb{N}$, $ia - (i-1)b < 2(i+1)a - 2ib$, or

$$2ib - (i-1)b < 2(i+1)a - ia.$$

Consequently, $(i+1)b < (i+2)a$, leaving $b < [(i+2)/(i+1)]a$, for all $i \in \mathbb{N}$. Hence, letting $i \rightarrow \infty$ implies $b \leq a$, a contradiction. \square

We now let $d(a, b)$ denote the (shortest) distance from the point (a, b) to the line $\{(x, y) \in \mathbb{R}^2 : x = y\}$. The following lemma establishes two important facts regarding the effect h has on this distance.

Lemma 4.2. *Let $(a, b) \in \mathbb{R}^2$ with $0 < a < b$ and $a/b \notin \mathbb{Q}$. If (a, b) also satisfies $b < 2a$, then $d(h(a, b)) = d(a, b)$. If (a, b) satisfies $b > 2a$ then $d(h(a, b)) < d(a, b)$.*

Proof. A direct calculation shows that for any point $(r, s) \in \mathbb{R}^2$ we have $d(r, s) = (\sqrt{2}/2)|r - s|$. For (a, b) with $b < 2a$, since $h(a, b) = (2a - b, a)$, we conclude that

$$d(h(a, b)) = \frac{\sqrt{2}}{2}|2a - b - a| = \frac{\sqrt{2}}{2}|b - a| = d(a, b).$$

For (a, b) with $b > 2a$, we again proceed by straightforward calculations, noting that

$$d(h(a, b)) = \frac{\sqrt{2}}{2}|b - 2a - b + a| = \frac{\sqrt{2}}{2}a.$$

On the other hand, $d(a, b) = (\sqrt{2}/2)(b - a)$. Since $\sqrt{2}/2a < \sqrt{2}/2(b - a)$ if and only if $b > 2a$, the result follows. \square

We are now in a position to prove the following, which completely describes the asymptotic behavior of Ducci sequences which begin with heterogeneous vectors in \mathbb{R}^3 .

Theorem 4.3. *Suppose $v = (a, b, c)$ is a heterogeneous vector with $a < b < c$. Then,*

$$\lim_{i \rightarrow \infty} \{h^i(g(0, b - a, c - a))\} = (0, 0).$$

Proof. As a notational convenience, let $h^i(g(0, b - a, c - a)) = (x_i, y_i)$ for each $i \geq 1$. By Lemma 4.1, we can form a subsequence of $\{(x_i, y_i)\}$ consisting of those terms where $y_i > 2x_i$. We denote this subsequence by $\{(p_i, q_i)\}$. Note that $d(p_i, q_i) > d(p_{i+1}, q_{i+1})$ for all $i \geq 1$ since $d(p_i, q_i) > d(h(p_i, q_i))$ for any $i \in \mathbb{N}$. Consequently, this is a decreasing sequence which is bounded below by zero, and therefore it follows that $\lim\{d(p_i, q_i) - d(p_{i+1}, q_{i+1})\} = 0$.

Referring back to the definition of h , one sees that for any $i \geq 0$, $y_i > y_{i+1}$. Furthermore, the sequence $\{y_i\}$ is also bounded below by zero. Hence, its limit exists, and we let $\lim\{y_i\} = c$. Note that if $c = 0$ then we are done, since for all $i \in \mathbb{N}$, $y_i > x_i$, and we can apply the squeeze theorem to conclude $\lim\{x_i\} = 0$ as well. The remainder of the proof will verify that in fact this must be the case.

Now, for $i \in \mathbb{N}$ given, let k be the smallest positive integer such that $h^k(p_i, q_i) = (P, Q)$ satisfies $Q > 2P$. By definition, it must be that $h^l(p_i, q_i) = (p_{i+l}, q_{i+l})$. If $k = 1$ then

$$d(p_i, q_i) - d(p_{i+1}, q_{i+1}) = \frac{\sqrt{2}}{2}(q_i - 2p_i)$$

follows immediately from Lemma 4.2. If $k > 1$, since $h^l(p_i, q_i)$ satisfies $q_i < 2p_i$ for all $1 \leq l < k$, it follows that

$$d(h(p_i, q_i)) = d(h^2(p_i, q_i)) = \dots = d(h^k(p_i, q_i)),$$

again by Lemma 4.2. Hence, the same conclusion regarding $d(p_i, q_i) - d(p_{i+1}, q_{i+1})$ applies in this case as well.

Thus, $\lim\{(\sqrt{2}/2)(q_i - 2p_i)\} = 0$, or equivalently, $\lim\{q_i - 2p_i\} = 0$. We know that $\lim\{q_i\} = \lim\{y_i\} = c$, so we conclude $\lim\{p_i\} = c/2$. But recall that $h(p_i, q_i) = (q_i - 2p_i, q_i - p_i)$. Furthermore, the sequence $\{y_i\}$ is decreasing. So $q_{i+1} \leq q_i - p_i$. Thus $\lim\{q_{i+1}\} \leq \lim\{q_i\} - \lim\{p_i\}$, or, $c \leq c - c/2$. It follows that $c \leq 0$. Since $c \geq 0$, we have that $c = 0$, as desired. \square

5. CONCLUSIONS AND FUTURE DIRECTIONS

The implication of Theorem 4.3, in conjunction with Lemma 3.2, is that for a given $v \in \mathbb{R}^3$ which is heterogeneous and for any $\varepsilon > 0$, there exists $H \in \mathbb{N}$ such that if $i \geq H$, then both entries of $g(f^i(v) - (K)_3)$ are positive and within ε of zero, where K is the smallest (necessarily

positive) entry of $f^i(v)$. Thus, we see that each entry of $f(f^i(v) - (K)_3) = f^{i+1}(v)$ is within ε of zero, and we conclude that $\lim\{f^i(v)\} = (0, 0, 0)$. Our results are then summarized in the following theorem.

Theorem 5.1. *Let $v \in \mathbb{R}^n$, n odd. If there exist $\alpha, x \in \mathbb{R}$, $\alpha \neq 0$, such that $\alpha(v + (x)_n) \in \mathbb{Q}^n$, then there exists $k \in \mathbb{N}$ such that $f^k(v)$ is a nontrivial periodic vector, and hence the Ducci sequence $\{f^i(v)\}$ is eventually periodic.*

If, on the other hand, $\alpha(v + (x)_n) \notin \mathbb{Q}^n$ for all $\alpha, x \in \mathbb{R}$, $\alpha \neq 0$, then $\{f^i(v)\}$ contains no periodic vectors, but approaches some periodic vector in \mathbb{R}^n asymptotically. In the case that $n = 3$, this periodic vector is the trivial periodic vector, $(0, 0, 0)$.

Finally, we can determine which of these two categories a given vector falls under by the method presented at the end of Section 2.

The result of Theorem 5.1 is perhaps surprising since it implies that two vectors which are arbitrarily close can have dramatically different asymptotic behavior. This is particularly apparent in the case $n = 3$, with one converging to the zero vector, and the other eventually becoming a nontrivial cyclic vector.

In fact, this behavior for $n = 3$ is distinct from that exhibited by the example in \mathbb{R}^7 given by [9] of a vector having a Ducci sequence which approaches, but never actually contains, a nontrivial periodic vector. The existence of such a vector is not unique to \mathbb{R}^7 ; indeed, it is not hard to find an example for each of \mathbb{R}^9 and \mathbb{R}^{15} , for instance. One is then led quite naturally to the question of whether the situations in \mathbb{R}^5 and \mathbb{R}^6 agree with the result in \mathbb{R}^3 or agree with the result of [9] (note that as observed above, the \mathbb{R}^4 case has been resolved by [32], among others. Generalizing further, as shown in [41], this problem has also been resolved for n any power of 2). One difference that is apparent between the cases $n = 5$ and $n = 6$ and those of \mathbb{R}^7 , \mathbb{R}^9 , and \mathbb{R}^{15} is that in the latter cases, the relevant nontrivial periodic vectors all had a period of length one, up to rotation. It is not hard to verify that there are no such nontrivial periodic vectors for $n = 5$, and for $n = 6$ the only such vector (again, up to rotation) is $(0, 1, 1, 0, 1, 1)$, which is simply two copies of $(0, 1, 1)$ in \mathbb{R}^3 . However, one should be careful not to draw too strong of conclusions from our observations, as the periodic

vector approached by our heterogeneous vector in \mathbb{R}^9 is actually $(0, 1, 1, 0, 1, 1, 0, 1, 1)$ (three copies of $(0, 1, 1)$).

It is worth noting that it becomes increasingly difficult to generalize the method used for \mathbb{R}^3 to higher dimensions due to the increase in complexity of the equations. The number of possible relative magnitudes among adjacent entries grows exponentially with vector length, and the number of possible relative magnitudes among all entries grows factorially. Hence, the simple piecewise-defined functions that we used above quickly grow in complexity. Furthermore, matters become even murkier since the ordering of a vector $v \in \mathbb{R}^n$'s entries does affect the magnitudes of $f(v)$'s elements for $n \geq 4$.

Finally, another problem that remains to be answered in future investigations is finding a criterion for discerning which vectors of even length yield an eventually periodic Ducci sequence. As stated in Corollary 2.7, heterogeneity is a necessary and sufficient condition for this property on vectors of odd length. Finding an analogous criterion for vectors of even length will lend much to our understanding of this problem.

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