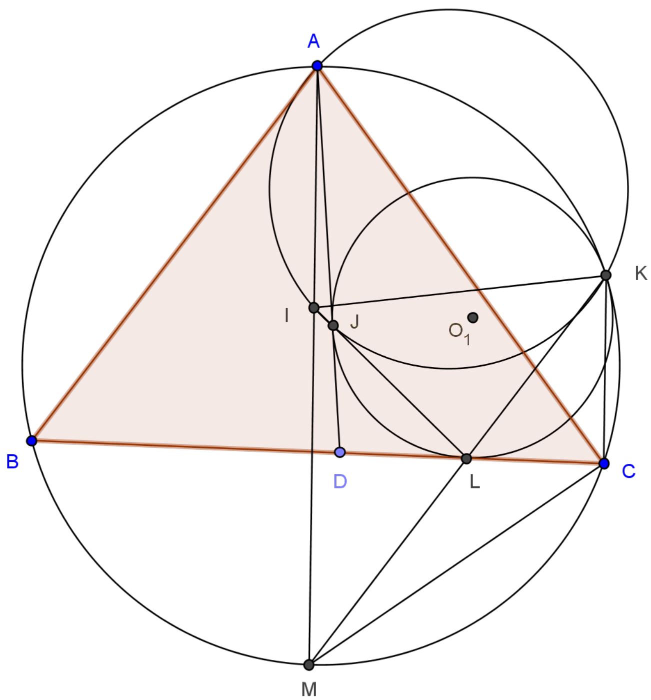
Sawayama Thebault's Theorem

1 Sawayama Thebault's Theorem

Given a triangle ABC, construct its circumcircle T. Now take any point D on side BC and draw line AD. Now construct the 2 circles that are internally tangent to T, BC, and AD, with centers O_1 and O_2 respectively. Let I be the incenter of ABC. O_1, O_2, and I are collinear.

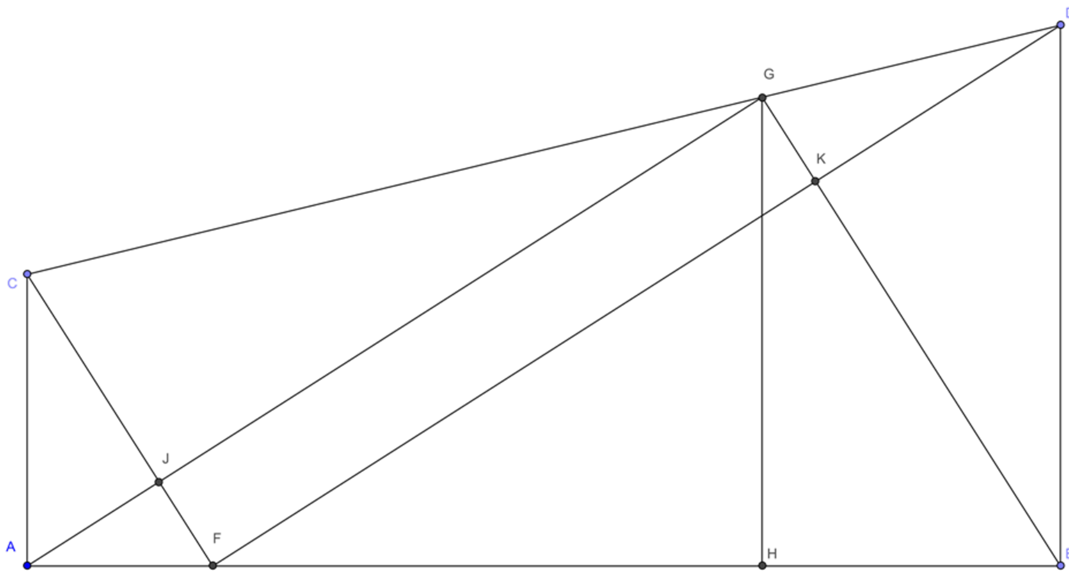
In order to prove this, we will first need the following 2 lemmas. The proof of the first lemma is due to Oleg Golberg.

Lemma 1: Let O_1 be tangent to T at K, BC at L, and AD at J. I, L, and J are collinear.



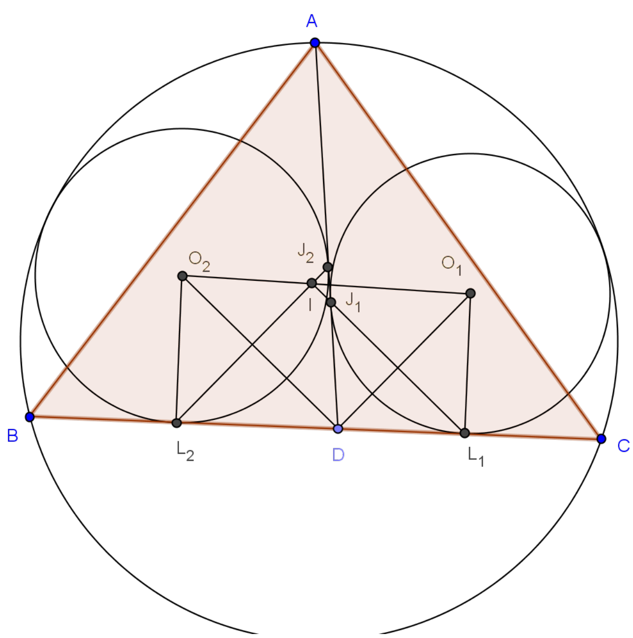
Proof: Let LI intersect O_1 at J'. We will show that J' and J coincide. Let KL intersect T for a second time at M. We can see from homothety that M is the midpoint of arc BC in T. Similarly, from homothety we can see that KL and KM subtend arcs of equal angles in O_1 and O respectively. We now have $\angle LJ'K = \angle MAK = \angle IAK = 180^\circ - \angle KJ'I$, which tells us that A, I, J', K are four cyclic points. Now we notice that $\angle MCB = \angle MBC = \angle MKC \rightarrow \triangle MLC \sim \triangle MCK \rightarrow MC^2 = ML * MK$. Noting the well known fact that $MI = MC$, we have $MI^2 = ML * MK \rightarrow \triangle MIL \sim \triangle MKI \rightarrow \angle MIK = \angle MLI \rightarrow \angle KLI = \angle KIA = \angle KJ'A$, which tells us that AJ' is tangent to O_1 as desired. ■

Lemma 2: Let ABCD be a trapezoid with $AC \parallel BD$, $AC \perp AB$, and $BD \perp AB$. Let there exist a point F on AB such that $\angle CFD = 90^\circ$. Draw a perpendicular to CF from A, and let it intersect CF and CD at J and G' respectively. Similarly, draw a perpendicular to DF from B, and let it intersect BF and CD at K and G'' respectively. G' and G'' coincide at a point called G.



Proof: Let $\angle DFB = \theta$. Now we notice that $G''K \parallel CF \rightarrow \frac{DG''}{G''C} = \frac{DK}{KF} = \frac{DK}{KB} * \frac{KB}{KF} = (\frac{DB}{FB})^2 = tan^2(\theta)$. Similarly, we notice that $JG' \parallel DF \rightarrow \frac{DG'}{G'C} = \frac{FJ}{JC} = \frac{FJ}{JA} * \frac{JA}{JC} = (\frac{FA}{AC})^2 = cot^2(90^\circ - \theta) = tan^2(\theta)$. It can now be seen that G' and G'' coincide as they are in the same location on CD.■

Extensions: It can now be seen that $\frac{DG}{GC} = tan^2(\theta)$. Now we note that $\frac{CD}{CG} = 1 + tan^2(\theta) = sec^2(\theta) \rightarrow \frac{CG}{CD} = cos^2(\theta) \rightarrow \frac{GD}{CD} = 1 - \frac{CG}{CD} = sin^2(\theta)$. Let H be the foot of the perpendicular from G to AB. We notice that $GH = \frac{CG}{CD} * BD + \frac{GD}{CD} * AC = AC * cos^2(\theta) + BD * sin^2(\theta)$.

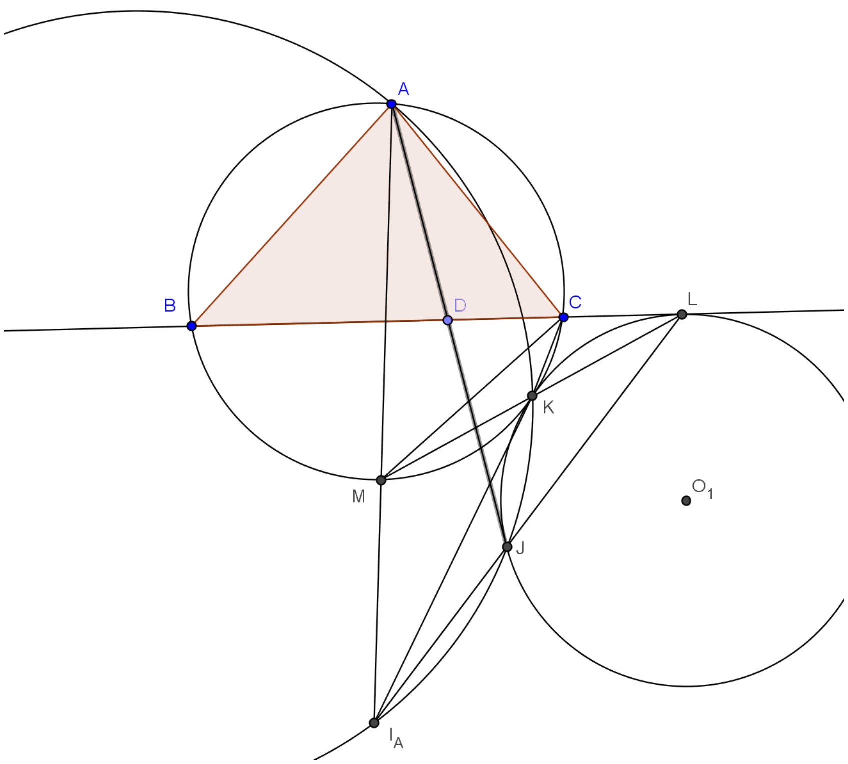


Sawayama Thebault's Theorem Proof: Let O_i be tangent to AD and BC at J_i and L_i respectively for $i = 1, 2$. Now we note that $\angle O_1 D O_2 = 90°$ as $O_1 D$ and $O_2 D$ are the internal and external angle bisectors of $\triangle ADB$ respectively. We also know that $O_i D \perp J_i L_i$ for $i = 1, 2$, which lets us identify $(O_1, L_1, D, L_2, O_2, I)$ as Lemma 2.■

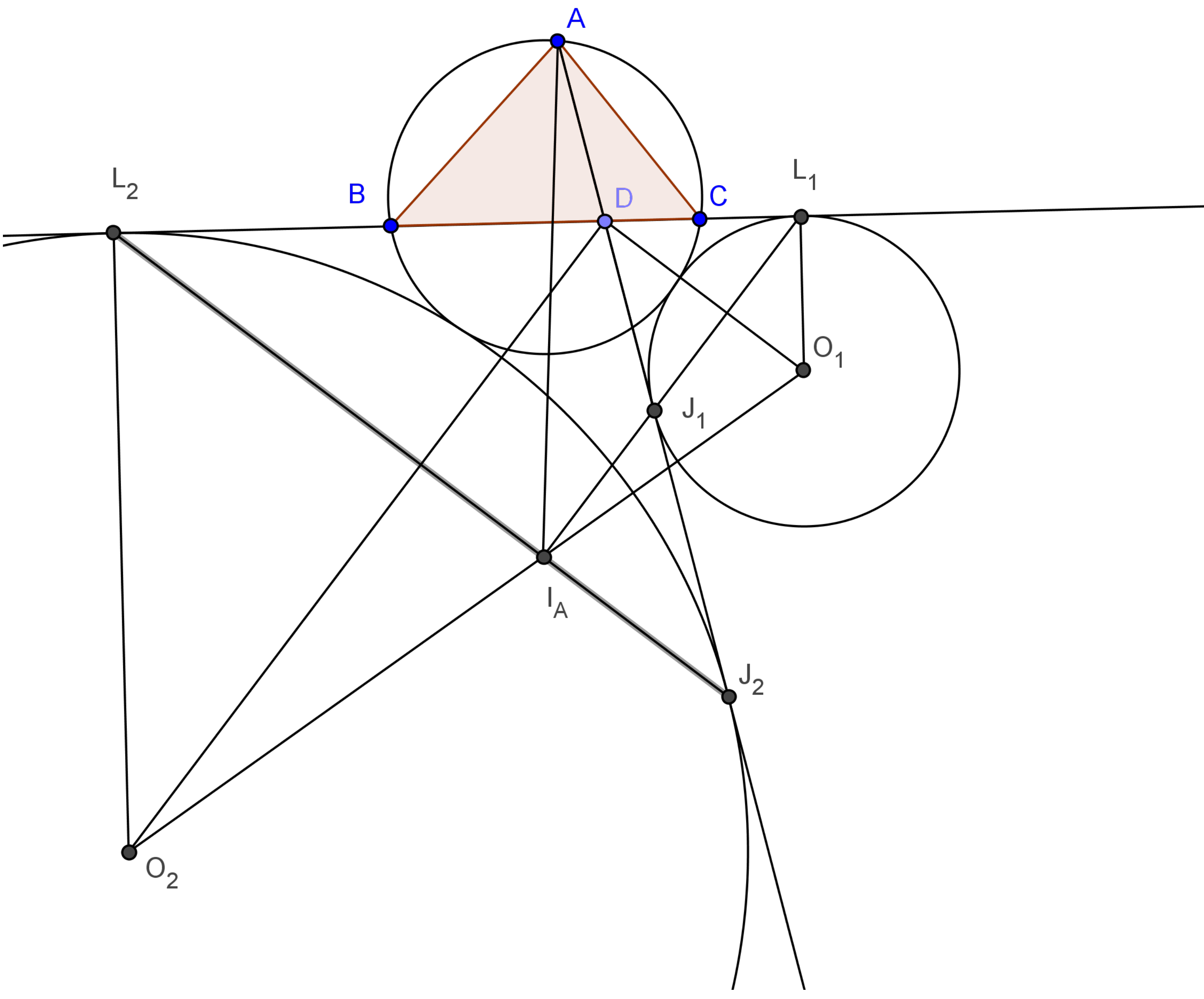
1.1 External Case

Given a triangle ABC, construct its circumcircle T. Now take any point D on side BC and draw line AD. Now construct the 2 circles that are externally tangent to T, BC, and AD, with centers O_1 and O_2 respectively. Let I_A be the excenter of ABC with respect to A. O_1, O_2, and I_A are collinear. Note that the following proof is almost identical to the proof of the internal case, so the reader may wish to attempt to prove this without first reading ahead.

Lemma 3: Let O_1 be tangent to T at K, BC at L, and AD at J. I_A, L, and J are collinear.



Proof: Let LI intersect O_1 at J'. We will show that J' and J coincide. Let KL intersect T for a second time at M. We can see from homothety that M is the midpoint of arc BC in T. Similarly, from homothety we can see that KL and KM subtend arcs of equal angles in O_1 and O respectively. We now have $\angle LJ'K = \angle MAK = \angle I_A AK = 180° - \angle KJ'I_A$, which tells us that A, I_A, J', K are four cyclic points. Now we notice that $\angle MCL = 180° - \angle MCB = 180° - \angle MBC = \angle MKC \rightarrow \triangle MLC \sim \triangle MCK \rightarrow MC^2 = ML * MK$. Noting the well known fact that $MI_A = MC$, we have $MI_A^2 = ML * MK \rightarrow \triangle MI_A L \sim \triangle MKI_A \rightarrow \angle MI_A K \angle MLI_A \rightarrow \angle KLI_A = \angle KI_A A = \angle KJ'A$, which tells us that AJ' is tangent to O_1 as desired. ■



External Case Proof: Let O_i be tangent to AD and BC at J_i and L_i respectively for $i = 1, 2$. Now we note that $\angle O_1DO_2 = 90°$ as O_1D and O_2D are the internal and external angle bisectors of $\triangle BDJ_1$ respectively. We also know that $O_iD \perp J_iL_i$ for $i = 1, 2$, which lets us identify $(O_1, L_1, D, L_2, O_2, I_A)$ as Lemma 2.$\blacksquare$

1.2 Excercises

3.1a) In the internal case, let D be such that circles O_1, O_2, and the incircle have the same radius. Prove that D is the point of tangency of BC with the excircle of vertex A.

3.1b) In the external case, let D be such that circles O_1, O_2, and the excircle have the same radius. Prove that D is the point of tangency of BC with the incircle.