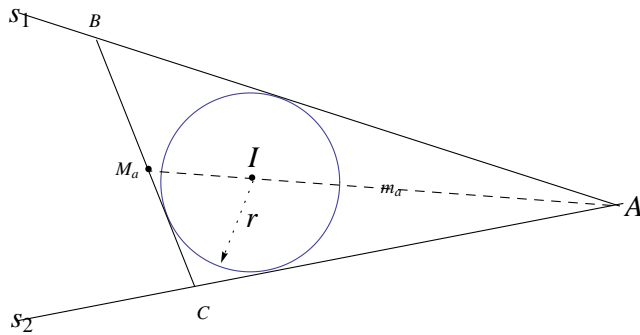


René Sperb , Dec. 1 , 2014

Triangle with given r , A , m_a



Analysis :

We have to find the points B , C such that the midpoint M_a has distance m_a from A .

To this end we let B and C move on the tangents s_1 and s_2 and determine the locus l of the corresponding midpoints M_a . Then we have to construct the point of intersection of this locus with the circle c_A of radius m_a around its center A .

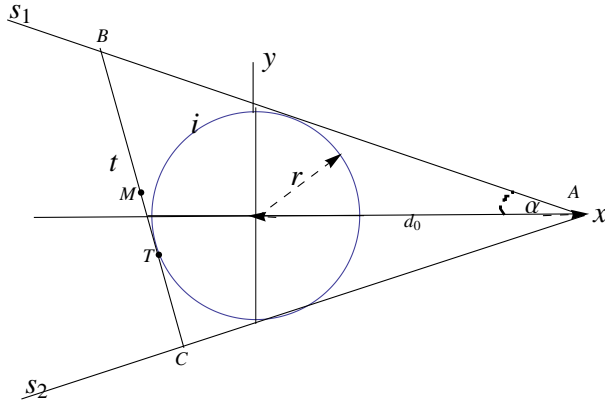
It will be shown that l is a hyperbola with center I and asymptotes parallel to s_1 and s_2 .

The intersection of l and c_A can in this case be constructed with compass and straightedge .

We will see that this problem can be transformed into the problem of intersecting a circle and a straight line.

To do so there are some calculations in analytic geometry required.

(1) Determination of the locus l :



Setting $\alpha = A/2$ and $m = \tan \alpha$ the tangents to the incircle may be written as $y = \pm m(x - d_0)$ and the tangent to i joining B and C as

$$x_1 x + y_1 y = r^2,$$

where (x_1, y_1) are the coordinates of T .

The coordinates of $B(x_2, y_2)$ and $C(x_3, y_3)$ are easily found from a linear system.

The coordinates of M are then

$$\left(\frac{1}{2}(x_2 + x_3), \frac{1}{2}(y_2 + y_3) \right).$$

If one sets

$$x_1 = r \cos t, \quad y_1 = r \sin t$$

one finds after routine calculation and some simplification that the coordinates of M are given by

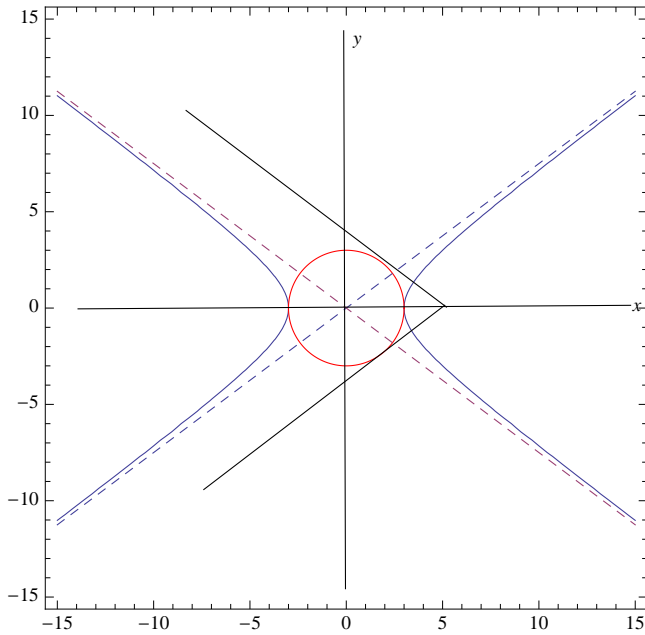
$$x = \frac{r(d_0 + r \cos t)}{r + d_0 \cos t}, \quad y = \frac{r^2 \sin t}{r + d_0 \cos t}.$$

From this one can deduce that the locus of M is the hyperbola

$$\frac{x^2}{r^2} - \frac{y^2}{\frac{r^4}{d_0^2 - r^2}} = 1.$$

The asymptotes are therefore parallel to the tangents s_1, s_2 .

The situation is as shown in the next figure :



(2) Intersection of hyperbola and circle

Of course the general case cannot be reduced to a quadratic equation. But in the present case we have the circle

$$(x - d_0)^2 + y^2 = m_a^2$$

and the hyperbola

$$\frac{x^2}{r^2} - \frac{y^2}{\frac{r^4}{d_2^2}} = 1,$$

where $d_2 = \sqrt{d_0^2 - r^2}$.

The intersection of the two curves requires the solution of the quadratic equation

$$(a) \quad x^2 \left(1 + \left(\frac{r}{d_2}\right)^2\right) - 2d_0x + d_0^2 - m_a^2 + \frac{r^4}{d_2^2} = 0.$$

We now compare this quadratic equation with the equation we would get if we have to intersect the circle

$$x^2 + y^2 = R^2$$

and the line

$$y = m x - q.$$

This leads to the equation

$$(b) \quad x^2 (1 + m^2) - 2 m q x + q^2 - R^2 = 0$$

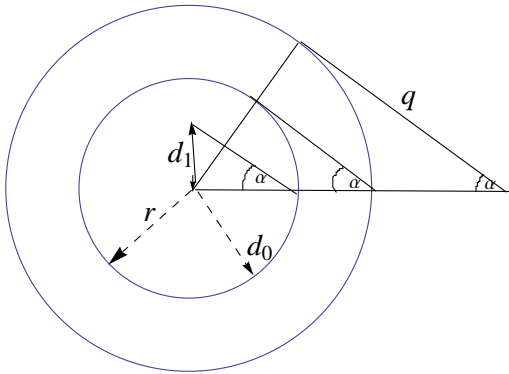
The comparison of (a) and (b) shows that if we select

$$m = \tan \alpha, q = \frac{d_0}{\tan \alpha}, d_1 = r \tan \alpha, \text{ and}$$

$$R = \sqrt{q^2 + d_1^2 + m_a^2 - d_0^2},$$

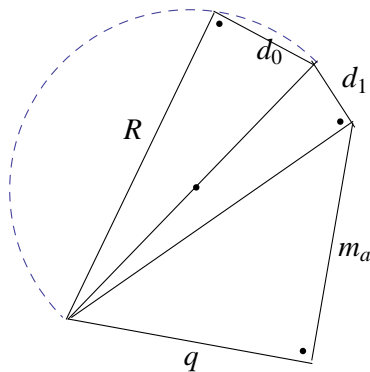
then the two equations (a) and (b) are equivalent.

The quantities d_0 , d_1 , q can be read from the next figure



(3) Construction of $R = \sqrt{q^2 + d_1^2 + m_a^2 - d_0^2}$:

The principle of construction, based on Pythagoras's theorem is shown in the next figure:



The final construction with all the important elements is summarized in the next figure

