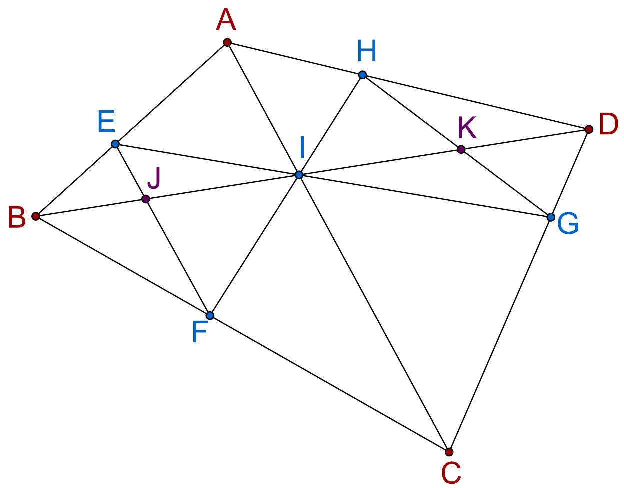
1 Kung's Lemma

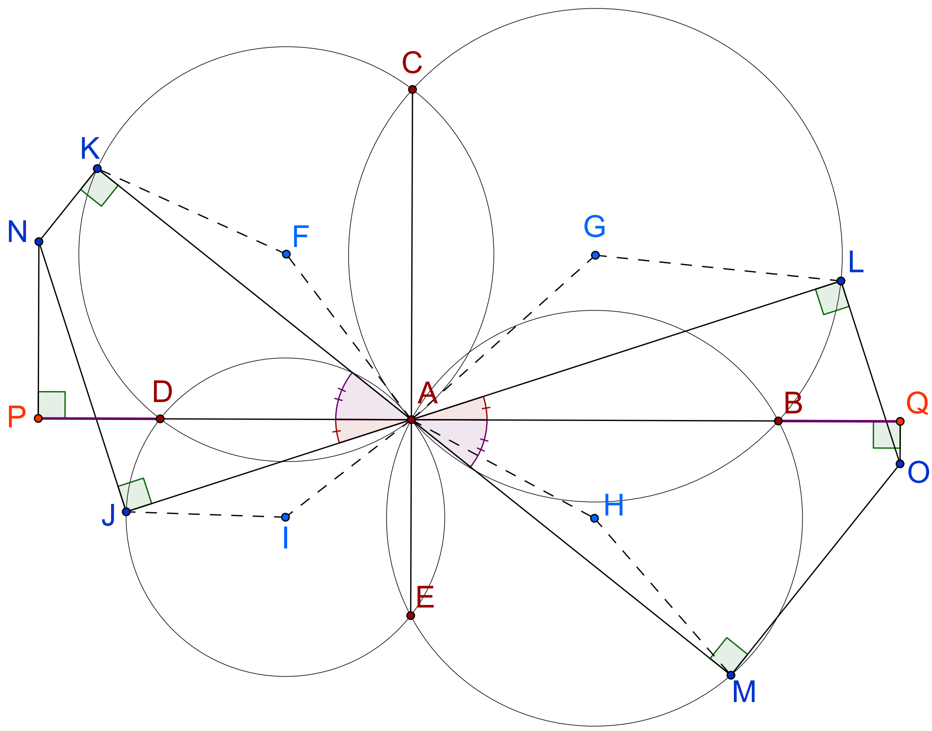
An alternative proof by Jack D'Aurizio (elianto84@gmail.com).



In the shown configuration,

$$\frac{1}{IJ} - \frac{1}{IB} = \frac{1}{IK} - \frac{1}{ID}$$

holds. We note that, without loss of generality, we can assume AC perpendicular to BD (by taking an affine transformation having BD as a line of fixed points). By inverting respect a circle centered in I we obtain the following configuration:



having to prove that $PD = BQ$ holds.
By choosing the following parametrization

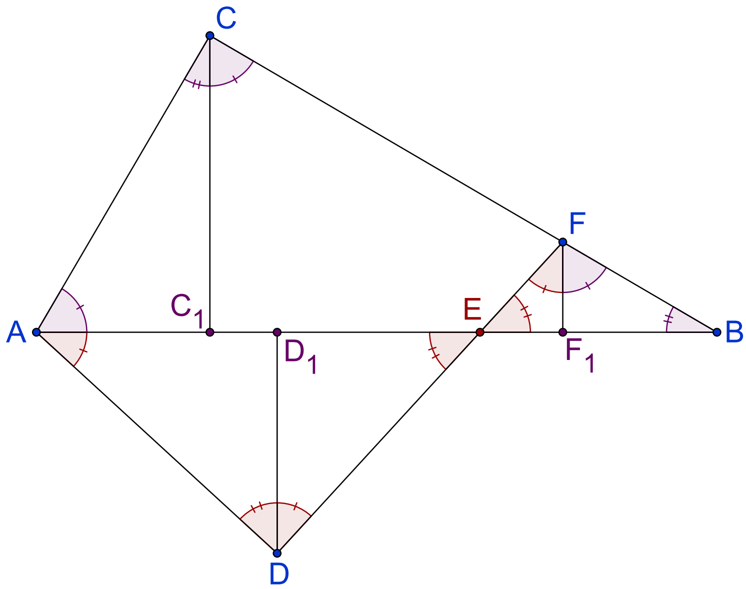
$$AB = x,\ AC = y,\ AD = z,\ AE = w,\ \widehat{BAL} = \theta,\ \widehat{MAQ} = \phi$$

we have

$$AL = \sqrt{x^2 + y^2}\,\cos(\arctan(y/x) - \theta) = x\cos(\theta) + y\sin(\theta)$$

$$AM = \sqrt{x^2 + w^2}\,\cos(\arctan(w/x) - \phi) = x\cos(\phi) + w\sin(\phi)$$

So, let's consider what happens in the next configuration,
where AC is perpendicular to BC and AD to DF:



Assuming $AC = u, AD = v, \widehat{BAC} = \theta, \widehat{DAB} = \phi$ we have

$$AB = \frac{u}{\cos(\theta)} \qquad AE = \frac{v}{\cos(\phi)}$$

$$EB = \frac{u\cos(\phi) - v\cos(\theta)}{\cos(\theta)\cos(\phi)}$$

so, by applying the sine law to the triangle EFB we obtain:

$$EF_1 = EF\cos(\phi) = \frac{EF}{EB} \cdot \frac{u\cos(\phi) - v\cos(\theta)}{\cos(\theta)} = \frac{u\cos(\phi) - v\cos(\theta)}{\sin(\theta)\cos(\phi) + \sin(\phi)\cos(\theta)}$$

If $u = x\cos(\theta) + y\sin(\theta)$ and $v = x\cos(\phi) + w\sin(\phi)$ as shown before,
it's extremely simple to note that neither

$$(AE - x)$$

nor

$$EF_1$$

depend on x, proving the theorem.